

Why certain Tannaka groups attached to abelian varieties are almost connected

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1 Introduction

Let k denote an algebraically closed field which either is the field \mathbb{C} of complex numbers or the algebraic closure of a finite field κ . Let Y be a smooth projective variety over k . Once we have fixed a point on Y we have the Albanese mapping

$$f : Y \longrightarrow Alb(Y) ,$$

with the induced mappings $f_i : S^i(Y) \rightarrow Alb(Y)$ from the symmetric powers of Y to the Albanese variety. The images are algebraic subvarieties in $Alb(Y)$, since the Albanese mapping is a proper morphism. In the simplest case, where Y is a curve, the study of the images of these maps leads to the well known Brill-Noether theory.

For higher dimensional varieties Y the Albanese mappings f_i are hardly understood and analogs of Brill-Noether theory usually brake down, if the degree i becomes larger than $\dim(Alb(Y))/\dim(Y)$. Already for surfaces this is unexplored. From the point of view of classical Brill-Noether theory the images of the f_i , which are the sums (i copies)

$$W_i(Y) = Y + \cdots + Y ,$$

suffice to recover many of the interesting properties of the curve Y and its line bundles. For higher dimensional varieties Y this is most likely not the case.

We therefore propose the following more sophisticated concept to replace and extend certain aspects of the ‘classical’ approach of Brill-Noether:

Choose some perverse sheaf on Y . The most natural and certainly most interesting candidate is the *IC*-perverse sheaf $IC_Y = \Lambda_Y[\dim(Y)]$, which by our assumptions on Y is just the constant sheaf on Y with coefficients in an algebraically closed field Λ of characteristic zero and shifted as a complex by the dimension of Y to make it into a perverse sheaf. Although we do not want to discuss this we

could also allow that Y is normal projective but not necessarily smooth, in which case we still dispose over the intersection cohomology sheaf IC_Y except that defining an Albanese morphism might not be obvious [VS]. But let us return to the case where Y is a smooth projective variety. Then on the Albanese variety $Alb(Y)$ the direct image $Rf_*(K)$ of the perverse sheaf $K = IC_Y$ is not necessarily a perverse sheaf. However by the decomposition theorem it decomposes into a direct sum of complexes which are semisimple perverse sheaves up to a complex shift. Such a complex will be called a semisimple complex for convenience.

In the case where Y is a curve, this direct image of course is an irreducible perverse sheaf with support in $f(Y)$ and will be denoted $K = \delta_Y$. We now consider all iterated convolution products of the perverse sheaves K and $-K := (-id_{Alb(Y)})^*(K)$ where the convolution is defined by $M * N = Ra_*(M \boxtimes N)$ for the group law of $X = Alb(Y)$ defined by $a : Alb(Y) \times Alb(Y) \rightarrow Alb(Y)$. Since the convolution product of two semisimple complexes is a semisimple complex, starting from $K = \delta_Y$ this constructs (denumerably) many new perverse sheaves on $Alb(Y)$ that arise as the irreducible perverse summands in these iterated convolution products. In the curve case for instance all the intersection cohomology sheaves $IC_{W_i(Y)}$ of the Brill-Noether varieties $W_i(Y)$ appear in this way. Indeed they arise as irreducible constituents in the i -th convolution product of δ_Y for $i = 1, \dots, g = \dim(Alb(Y))$. Moreover the natural Brill-Noether stratification of the singular variety $W_i(Y)$, defined by the well known subvarieties $W_i^r(Y)$, can be naturally recovered from the supports of the cohomology sheaves $\mathcal{H}^{2r-i}(IC_{W_i(Y)})$ of the sheaf complexes $IC_{W_i(Y)}$. So this approach contains all the relevant input for the classical Brill-Noether theory, but of course at the cost of introducing infinitely many more perverse sheaves in addition, which do not seem to be needed and also do not seem to be easy to understand at first glance. But this becomes not at all disturbing any more, once it is understood that the perverse sheaves constructed in this way can in some sense be interpreted as the irreducible representations of a reductive group $G(Y)$ over Λ .

In the case of a (non-hyperelliptic) curve Y of genus g this group $G(Y)$ turns out to have the commutator group $G(V) = Sl(2g - 2)$. So this commutator group has $g - 1$ fundamental representations, namely the alternating powers $\Lambda^i(V)$ of the $2g - 2$ dimensional standard representation $V = V(Y)$ of $G(V)$. Furthermore these fundamental representations correspond exactly to the $g - 1$ irreducible perverse sheaves $IC_{W_i(Y)}$ for $i = 1, \dots, g - 1$ already mentioned above. So in this sense we see how to recover the essential geometric objects $W_i(Y)$ from the generator δ_Y via group theory using the representation theory of $G(Y)$ by means of the tensor structure defined by the convolution product: They correspond to the fundamental

representations of $G(V)$ on the alternating tensor powers of V

$$\Lambda^i(V) \longleftrightarrow W_i(Y)$$

for $i \leq g-1$. This picture also incorporates Riemann-Roch and Serre duality in the sense that we have

$$\Lambda^{2g-2-i}(V) \cong \det(V) \otimes \Lambda^i(V)^* \longleftrightarrow W_{2g-2-i}(Y) = \kappa - W_i(Y)$$

for the Riemann constant $\kappa \in \text{Alb}(Y)$ corresponding to $\det(V)$.

For higher dimensional varieties such a perverse approach has the added merit that the convolution products K^{*i} of the perverse sheaf $K = \delta_Y$ with i factors remain to have a meaning beyond the critical bound $i > \dim(\text{Alb}(Y))/\dim(Y)$, whereas the images $f_i(Y) \subseteq \text{Alb}(Y)$ in contrast usually tend to be $\text{Alb}(Y)$ and therefore then simply are meaningless. Furthermore this new approach easily allows to extend the Brill-Noether construction in the following sense: To any irreducible perverse sheaf K on Y one may consider the direct image $Rf_*(K)$ on $\text{Alb}(Y)$ and study the tensor category defined by the convolution products of K in the same way. It turns out that once again one can define a reductive group $G(K)$ over Λ and a natural Λ -rational representation $V(K)$ of $G(K)$ generalizing the special case discussed above. This more general point of view then also allows to incorporate functoriality. Indeed suppose

$$g : Y_1 \longrightarrow Y_2$$

is a morphism between smooth projective varieties Y_1 and Y_2 . Then we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{g} & Y_2 \\ \downarrow f_1 & & \downarrow f_2 \\ \text{Alb}(Y_1) & \longrightarrow & \text{Alb}(Y_2) \end{array}$$

For any irreducible perverse sheaf K_1 , or more generally semisimple sheaf complex on Y_1 the direct image complex $K_2 = Rg_*(K_1)$ is a semisimple sheaf complex on Y_2 . So we can study the reductive group $G(K_1)$ defined by K_1 and also the group $G(K_2)$ defined by K_2 . These groups are related. For instance if $\text{Alb}(g)$ is surjective, then $G(K_2)$ is a closed subgroup of $G(K_1)$, and otherwise at least it is a subquotient. So this defines a contravariant functor

$$G(g) : G(K_2) \longrightarrow G(K_1) .$$

Hence $G(K_1)$ defines a bound for $G(K_2)$. Notice then, the fact that in general $K_2 = Rg_*(K_1)$ is not perverse is related to the size of the subgroup $G(K_2) \subseteq G(K_1)$. To understand this let us recall the general construction given in [KrW]. It relies on the construction of a tensor category and the mechanism of Tannaka duality for the abelian variety $X = Alb(Y)$ under consideration. The groups we are interested in are reductive quotient groups of a big profinite Tannaka group $\mathbf{G}(X)$. For this recall that to an abelian variety X over k we attached in [KrW] a pro-algebraic reductive supergroup $\mathbf{G}(X)$ defined over a certain algebraically closed coefficient fields Λ . For later use let us mention that it comes together with an epimorphism of groups

$$\mathbf{G}(X) \twoheadrightarrow \pi_1^{et}(\hat{X}, 0)(-1)$$

whose kernel $\mathbf{G}^1(X)$ contains the connected component $\mathbf{G}^0(X)$ of $\mathbf{G}(X)$, where $\mathbf{G}^0(X)$ is defined as the projective limit of the connected components of all algebraic quotient groups of $\mathbf{G}(X)$. Here $\pi_1^{et}(\hat{X}, 0)$ is the etale profinite fundamental group of the dual abelian variety \hat{X} . If k is the field of complex numbers, then this profinite supergroup in fact is a projective limit of reductive groups of Λ and our groups arise as algebraic quotient groups of this projective limit.

To define $\mathbf{G}(X)$ we assume $\Lambda = \overline{\mathbb{Q}}_l$ or \mathbb{C} , if $k = \mathbb{C}$, or we assume $\Lambda = \overline{\mathbb{Q}}_l$, if k is the algebraic closure of a finite field. Then the bounded derived category $D_c^b(X, \Lambda)$ can be shown to be a symmetric monoidal rigid Λ -linear tensor category. Define $D(X) = D_c^b(X, \Lambda)$ for $k = \mathbb{C}$, respectively for $char(k) \neq 0$ define $D(X)$ as the full symmetric monoidal rigid Λ -linear tensor category of complexes in $D_c^b(X, \Lambda)$ which are defined over a finite subfield of k . Here the tensor product on $D(X)$ is given by the convolution product $K * L$ of complexes K, L as defined above. Let $P(X)$ be the full subcategory of $D_c^b(X, \Lambda)$ of semisimple perverse sheaves in $D(X)$. Furthermore let $D^{ss}(X)$ be the full tensor subcategory of semisimple complexes in $D(X)$, i.e. those objects which are finite direct sums of objects in $P(X)$ up to a complex shift. Then the complexes in $D^{ss}(X)$, whose simple constituents have Euler characteristic zero, define a tensor ideal N_{Euler} in $D^{ss}(X)$. The quotient tensor category of $(D^{ss}(X), *)$ obtained after dividing by this tensor ideal N_{Euler} of ‘negligible’ complexes is a semisimple neutral super-Tannakian category $\overline{D}^{ss}(X)$ to be considered as a kind of ‘semi-classical limit’ of $(D^{ss}(X), *)$.

As a tensor category the quotient category $\overline{D}^{ss}(X)$ is isomorphic to a semisimple category $sRep_\Lambda(\mathbf{G}(X), \mu)$ of finite dimensional super representations over Λ , with some central constraint μ , of a pro-algebraic affine reductive supergroup $\mathbf{G}(X)$. The tensor subcategory generated by any object K in $\overline{D}^{ss}(X)$ therefore is an algebraic reductive supergroup over Λ ; and for $k = \mathbb{C}$ it is even known to be

a classical reductive group over Λ . These are the groups $G(K)$ considered above. Via Tannaka duality the complex K itself then defines a finite dimensional faithful representation $V(K)$ of the group $G(K)$

$$(G(K), V(K))$$

and this representation is irreducible, if K is an irreducible perverse sheaf on X .

Now let us return to the analysis of non-negligible higher perverse direct images for perverse sheaves K_1 on X (for simplicity we assume $X = Y_1$). In fact this is reflected by a canonical product decomposition of the pro-group $\mathbf{G}(X)$ into a geometric part $G(X)$ and a multiplicative group \mathbb{G}_m , where the irreducible representations of the multiplicative part \mathbb{G}_m keep track of the different perverse cohomology groups of a semisimple complex; i.e. $t \in \mathbb{G}_m$ acts on the i -th perverse cohomology ${}^p H^i(M)$ of a semisimple complex M via multiplication by t^i . In this sense the appearance of higher perverse cohomology for the direct image $Rg_*(K_1)$ gives rise to a decomposition of the restriction of the irreducible representation $V(K_1)$ of $G(K_1)$ into reducible representations under the pullback with respect to the natural homomorphism $G(K_2) \rightarrow G(K_1)$, that one can define for $K_2 = Rg_*(K_1)$ and the corresponding group $G(K_2)$. To make this more precise let $\bar{P}(X)$ denote the image of $P(X)$ in $\bar{D}^{ss}(X)$. Then, at least in the complex case $k = \mathbb{C}$, it can be shown that

$$\mathbf{G}(X) = \mathbb{G}_m \times G(X)$$

holds such that $sRep_\Lambda(\mathbf{G}(X), \mu) \cong sRep_\Lambda(\mathbb{G}_m, \mu) \otimes_\Lambda Rep(G(X))$, where $Rep(G(X))$ is equivalent as a tensor category to the Tannakian tensor subcategory $\bar{P}(X)$ of so called perverse multipliers in $\bar{D}^{ss}(X)$; for details we refer to [KrW]. The last statement is closely related to the following vanishing theorem of [KrW], [W]

Theorem 1. *Let $f : X \rightarrow Y$ be a homomorphism between complex abelian varieties and let K be a perverse sheaf on X . Then for ‘most’ characters χ of $\pi_1(X, 0)$ the direct image $Rf_*(K_\chi)$ of the character twist K_χ of K is a perverse sheaf on Y .*

In the formulation of this theorem χ is a character $\chi : \pi_1(X, 0) \rightarrow \mathbb{C}^*$ of the topological fundamental group $\pi_1(X, 0)$ of X , and K_χ denotes the perverse sheaf $K \otimes_\Lambda L_\chi$ where L_χ is the rank one étale local system on X attached to the character χ of the fundamental group $\pi_1(X, 0)$. For the definition of the notion ‘most’ we refer to [W, section 1]; if Y is the trivial abelian variety, it means: χ is in the complement of a suitable finite union of translates $\chi_i \cdot K(X_i)$ for nontrivial abelian subvarieties $X_i \subseteq X$, where $K(X_i)$ denotes the group of characters $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$

whose restriction to the subgroup $\pi_1(X_i, 0)$ of $\pi_1(X, 0)$ vanishes. See [KrW] for further details on this.

It is now an intriguing question how one can compute the groups $G(K)$. This seems to be quite difficult even in the case where $G(K) = G(Y)$ is defined by the IC-sheaf $K = IC_Y$. This difficulty comes from the fact that convolution products are not easy to compute in practice. Nevertheless it is reasonable to believe, at least in the case $K = IC_Y$, that the direct image complex $M = Rf_*(K)$ defined by the Albanese morphism f defines some small and distinguished representations of $G(Y)$ similar as in the curve case. This is a rather tempting assumption and at least plausible in the case where the Albanese morphism is a birational morphism. So, once the groups $G(Y)$ are understood to have a sufficiently easy structure so that their representations can be described by some ‘fundamental’ representations, this should help to compute the groups $G(Y)$. In fact an easy description exists for the representations of the Zariski connected component $G^0(Y)$ of the algebraic (super)group $G(Y)$ by the well known representation theory of H. Weyl describing irreducible representations in terms of highest weights. If the algebraic group $G(Y)$ is Zariski connected (or at least almost connected), then all irreducible representations of $G(Y)$ and also the group itself should be detectable by some ‘small’ fundamental representations or other distinguished small representations of $G(Y)$. Here comes into play the following

Conjecture. *The group $G^1(X)$ is equal to the ‘Zariski connected’ component $G^0(X)$ of the pro-algebraic supergroup $G(X)$.*

In fact for complex abelian varieties we have the

Theorem 2. *For complex abelian varieties the conjecture above is true.*

In fact the vanishing theorem 1 stated above allows to reduce the proof of this theorem 2 to the following statement: Perverse sheaves $K \cong K^\vee$ with the property $K^{*2} \cong d \cdot K$ and without negligible constituents are skyscraper sheaves on X ; in fact their support is contained in a finite set of torsion points of $X(\mathbb{C})$ and this statement can be proved by induction on the dimension of X . For the induction step we use reduction mod p to deduce the assertion from a similar statement for abelian varieties over finite fields. Over finite fields then as an essential ingredient we use a theorem of Drinfeld [Dr2] on the existence of compatible l -adic systems for perverse sheaves. Although the proof of the conjecture uses methods of characteristic $p > 0$, the reduction steps can not yet be carried over to the case of positive characteristic so far.

So taking for granted the results above, we then conclude that for $k = \mathbb{C}$ the group $\pi_0(G(Y))$ of Zariski connected components is an abelian group and isomorphic to a certain finite group H of torsion points on the abelian variety X . So considering the semisimple complex $M = Rf_*(K)$ on X defined for say $K = IC_Y$ and the Albanese mapping $f : Y \rightarrow X = Alb(Y)$, let us (for simplicity) assume that M is an irreducible perverse sheaf X . Then we can assume that the stabilizer of M , i.e. the abstract group H of closed points x of X such that $T_x^*(M) \cong M$ holds for the translation $T_x : X \rightarrow X$ by x , is finite (otherwise $G(Y)$ is trivial). If H is not zero, then we can replace X by the isogenous abelian variety X/H and thereby reduce to the case $H = \{0\}$; see section 2 and 3 and lemma 11. Now if H is zero, then the results of section 2 and our main result theorem 2 above in fact imply that the restriction of the irreducible representation of $G(Y)$ on $V(Y)$, defined by the irreducible perverse sheaf $M = Rf_*(K)$ using Tannaka duality, remains irreducible after restriction to the connected component $G^0(Y)$. In other words: $V(Y)$ defines a highest weight representation of the semisimple algebraic group $G^0(Y)$. So if we expect that this natural representation is in some sense a distinguished small representation of $G^0(Y)$, then in the nonadjoint cases one has a good chance to determine the group $G^0(Y)$ or at least its commutator group from the knowledge of the decomposition of the convolution tensor square $M * M$ of the perverse sheaf M . In the curve case (using [KrW3]) this in fact easily determines $G(Y)$, and this kind of argument considerably simplifies the approach taken in [BN]. For another nontrivial example, where this strategy determines $G(Y)$, see [KrW2]; here Y is a threefold.

Finally let me briefly indicate the reason, why the Tannaka groups $G(Y)$ respectively their profinite counterparts are almost connected (in these sense of theorem 2 above). Indeed this comes from the fact that the underlying group structure of X is abelian, so that using Fourier transform the assertion can be converted into some equivalent but now rather elementary assertion on the existence of many torsion points (lemma 17). To reveal this we use an argument involving reduction mod p , as already mentioned, and the Cebotarev density theorem (see section 13). It seems very likely that one should be able to prove this directly by the theory of D -modules in characteristic zero. However, since our proof finally aims for the case of varieties over fields of positive characteristic also, we did not intend to give such a proof. Nevertheless it would be interesting to have a purely analytic argument.

Finally, in the appendix we discuss perverse sheaves with vanishing Euler characteristic on abelian varieties with a similar technique.

2 Isogenies

An isogeny $f : X \rightarrow Y$ of abelian varieties over k with kernel F

$$0 \rightarrow F \rightarrow X \rightarrow Y \rightarrow 0$$

factorizes into a composite of an inseparable isogeny and a separable finite etale isogeny, and the degree of the latter will be denoted $\deg_s(f)$. Attached to f there is an embedding of groups $G(f) : G(Y) \hookrightarrow G(X)$ (see [KrW]).

Lemma 1. *The inclusion $G(f) : G(Y) \rightarrow G(X)$ identifies $G(Y)$ with a normal subgroup of finite index. This index is $\deg_s(f) = [G(X) : G(Y)]$ and there exists a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^1(Y) & \longrightarrow & G(Y) & \longrightarrow & \pi_1^{et}(\hat{Y}, 0)(-1) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow G(f) & & \downarrow \\ 0 & \longrightarrow & G^1(X) & \longrightarrow & G(X) & \longrightarrow & \pi_1^{et}(\hat{X}, 0)(-1) \longrightarrow 0 \end{array}$$

Proof. The functors

$$R = f_* : \text{Perv}(X, \Lambda) \rightarrow \text{Perv}(Y, \Lambda) \quad , \quad I = f^* : \text{Perv}(Y, \Lambda) \rightarrow \text{Perv}(X, \Lambda)$$

are adjoint functors preserving perversity, if f is finite and etale ([BBD] cor. 2.2.6) or if f is completely inseparable. Hence

$$\text{Hom}_{G(X)}(I(K), L) = \text{Hom}_{G(Y)}(K, R(L)) .$$

Notice that f^* and f_* preserve semisimplicity by corollary 1 below respectively by the decomposition theorem. In the Tannakian context of representations of supergroups these functors I and R correspond to induction and restriction.

Let $\pi_1(X, 0)$ denote the topological fundamental group for $k = \mathbb{C}$ respectively the etale fundamental group for fields k of characteristic > 0 . Then the group $F(k)$, whose order is $\deg_s(f)$, can be identified with the cokernel of the homomorphism $\pi_1(f) : \pi_1(X, 0) \rightarrow \pi_1(Y, 0)$. Hence the Pontryagin dual F^* of F can be identified with the group of characters $\chi : \pi_1(Y, 0) \rightarrow \Lambda^*$ vanishing on the image of $\pi_1(X, 0)$. For a ‘continuous’ character $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$ and the corresponding local system

L_χ on X , recall that $K_\chi = K \otimes_\Lambda L_\chi$ for $K \in D(X)$ denotes the twisted complex. Then $RI(K) = Rf_*(\Lambda_X) \otimes_\Lambda K$ and $Rf_*(\Lambda_X) = \bigoplus_{\chi \in F^*} K_\chi$ imply

$$RI(K) = \bigoplus_{\chi \in F^*} K_\chi .$$

By $\dim(K_\chi) = \dim(K)$ (see [KrW, cor.10]) this computes the categorial dimension $\dim(RI(K))$. As a tensor functor $R = Rf_*$ preserves categorial dimensions

$$\dim(I(K)) = \deg_s(f) \cdot \dim(K) .$$

To prove lemma 1 the pro-algebraic groups $G(X)$ and $G(Y)$ can be replaced by algebraic quotient supergroups, so by abuse of notion these also will be denoted $G(X)$ and $G(Y)$. An algebraic supergroup $G(X)$ is a triple $(G_c(X), \mathfrak{g}(X)_-, Q_X)$, where $G_c(X)$ is a classical algebraic reductive affine algebraic group over Λ considered as a closed subgroup of the supergroup $G(X)$ and \mathfrak{g}_- is the finite dimensional odd part of the super Lie algebra of $G(X)$. See [KrW, appendix]. The restriction functor

$$R_X : sRep_\Lambda(G(X)) \longrightarrow sRep_\Lambda(G_c(X))$$

between the categories of finite dimensional super representations of $G(X)$ and $G_c(X)$ over Λ admits an adjoint induction functor

$$I_X : sRep_\Lambda(G_c(X)) \longrightarrow sRep_\Lambda(G(X))$$

given by $I_X(L) = L \otimes_{U(Lie(G_c(X)))} \Lambda^\bullet(\mathfrak{g}(X)_-)$, for L in $sRep_\Lambda(G_c(X))$. Similarly for Y instead of X . The isogeny f induces an embedding $G(f) : G(Y) \hookrightarrow G(X)$ and hence a closed embedding $G_c(Y) \hookrightarrow G_c(X)$ of the reductive algebraic subgroups of the supergroups. This defines an obvious restriction functor

$$R_c : sRep_\Lambda(G_c(X)) \longrightarrow sRep_\Lambda(G_c(Y)) .$$

The adjoint functor for the composed functor $R_Y \circ R$ (defining the restriction $G_c(Y) \hookrightarrow G(Y) \hookrightarrow G(X)$) is $J = I \circ I_Y$. The restriction functors related to the next diagram commute: $R_Y \circ R = R_c \circ R_X$

$$\begin{array}{ccc} G(Y) & \hookrightarrow & G(X) \\ \uparrow & & \uparrow \\ G_c(Y) & \hookrightarrow & G_c(X) \end{array}$$

This implies $\text{Hom}_{G(X)}(J(1), K) = \text{Hom}_{G_c(Y)}(1, R_Y \circ R(K))$ and $\text{Hom}_{G_c(Y)}(1, R_Y \circ R(K)) = \text{Hom}_{G_c(Y)}(1, R_c \circ R_X(K))$.

Let now L denote some irreducible object in $s\text{Rep}_\Lambda(G_c(X))$. Since $s\text{Rep}_\Lambda(G(X))$ is a semisimple category, we find an irreducible summand K of $I_X(L)$ so that $R_X(K)$ contains L and $\text{Hom}_{G_c(Y)}(1, R_c(L)) \subseteq \text{Hom}_{G_c(Y)}(1, R_c \circ R_X(K))$ holds. Hence $\text{Hom}_{G(X)}(J(1), K) = 0$ implies $\text{Hom}_{G_c(Y)}(1, R_c(L)) = 0$. Now $J(1)$ contains only finitely many irreducible subrepresentations, and these together have only finitely many irreducible representations L in their restriction to $G_c(X)$. Hence there exist only finitely many isomorphism classes of irreducible representations L of $G_c(X)$ whose restriction $R_c(L)$ to the subgroup $G_c(Y)$ contains the trivial representation. Of course the analogous assertion then also holds for the connected components $G_c^0(X)$ and $G_c^0(Y)$. Let Σ denote the finite set of highest weights of these finitely many representations. Since $\text{char}(\Lambda) = 0$, we claim that classical invariant theory in this setting now implies $G_c^0(X) = G_c^0(Y)$. Indeed, the coordinate ring $\Lambda[G_c^0(X)/G_c^0(Y)] = \Lambda[G_c^0(X)]^{G_c^0(Y)}$ of the quotient variety $G_c^0(X)/G_c^0(Y)$ is affine, since $G_c^0(Y)$ is a reductive algebraic group [M]. For irreducible $G_c^0(X)$ -subcomodules L in the coordinate ring $\Lambda[G_c^0(X)]$ then $L^{G_c^0(Y)} = 0$ holds for $\alpha \notin \Sigma$, if L defines an irreducible representation of $G_c^0(X)$ of highest weight α . This implies $\dim_\Lambda(\Lambda[G_c^0(X)/G_c^0(Y)]) < \infty$, since up to isomorphism any irreducible comodule L occurs with finite multiplicity in the coordinate ring $\Lambda[G_c^0(X)]$. Since the algebraic varieties $G_c^0(X)$ and hence $G_c^0(X)/G_c^0(Y)$ are Zariski connected, the Λ -algebra $\Lambda[G_c^0(X)/G_c^0(Y)]$ therefore is obtained by adjoining nilpotent elements to Λ . Hence $\Lambda[G_c^0(X)/G_c^0(Y)] = \Lambda$ by [GIT, thm. 1.1]; see also page 5 of loc. cit. This proves our claim $G_c^0(X) = G_c^0(Y)$. Hence the reductive group $G_c(Y)$ has finite index in the reductive group $G_c(X)$. Therefore the restriction functor R_c , defined above, admits an adjoint induction functor I_c . This implies

$$I = I_- \circ I_c$$

where I_c is the superinduction functor $I_-(M) = M \otimes_{U(\text{Lie}(G(Y)))} \Lambda^\bullet(\mathfrak{g}(X)_-/\mathfrak{g}(Y)_-)$. For all $G(Y)$ -modules K the super dimension of $I_-(K)$ is zero unless $\mathfrak{g}(X)_- = \mathfrak{g}(Y)_-$ (in view of $G_c^0(X) = G_c^0(Y)$ this is equivalent to $G^0(Y) = G^0(X)$). This also holds for the functor $I = I_- \circ I_c$ being inherited from I_c . Now on the other hand, the superdimension $\dim(K)$ of a nontrivial simple object in a semisimple algebraic tensor category is never zero. This implies $\dim(I(K)) = \deg_s(f) \cdot \dim(K) \neq 0$. Together with the observation from above this implies

$$G^0(Y) = G^0(X) .$$

Hence $I = I_c$ and $\dim(I(K)) = [\pi_0(G(X)) : \pi_0(G(Y))] \cdot \dim(K)$, so that we get

$$[\pi_0(G(X)) : \pi_0(G(Y))] = \deg_s(f) .$$

To show that $G(Y)$ is a *normal* subgroup of $G(X)$ it now suffices to show

$$[\pi_0(G(X))^{ab} : \pi_0(G(Y))^{ab}] \geq \deg_s(f) .$$

For this observe that any $x \in F(k)$ defines a one dimensional character of $G(X)$, whose restriction to $G(Y)$ is the unit object $Rf_*(\delta_x) = \delta_0$ and defines the trivial representation of $G(Y)$. Thus shows $[\pi_0(G(X))^{ab} : \pi_0(G(Y))^{ab}] \geq \#F(k)$, and immediately proves the claim. \square

$G(X)$ acts on $G(Y)$ by conjugation. For a representation K of $G(Y)$ and $g \in G(X)$ let K^g denote the representation of $G(Y)$ twisted by conjugation with g . Up to isomorphism K^g depends only on the coset $g \in G(X)/G(Y)$. Recall that $G(Y)$ is a normal subgroup of $G(X)$. By Mackey's lemma the conjugates K^g of an irreducible representation K of $G(Y)$ are the constituents of the module $R(I(K))$. Hence they are all isomorphic to simple modules K_χ obtained from K by a character twist using

$$\bigoplus_{g \in G(X)/G(Y)} K^g = R(I(K)) \cong \bigoplus_{\chi} K_\chi .$$

The elements χ in the sum on the right side correspond to the characters in the Pontryagin dual F^* of F . The elements in $G(X)/G(Y)$ are described by $\pi_1^{et}(\hat{X}, 0)(-1)/\pi_1^{et}(\hat{Y}, 0)(-1) = \hat{F}(-1)$ for the Cartier dual \hat{F} of F , and $\hat{F}(-1)$ can be identified with the Pontryagin dual F^* . This is obvious for $\text{char}(k) = 0$ and left as an exercise for $\text{char}(k) > 0$.

Lemma 2. *For a finite group H of torsion points in X consider $f : X \rightarrow X' = X/H$. Then for a perverse sheaf K on X the following assertions are equivalent:*

1. $T_x^*(K) \cong K$ holds for all $x \in H$.
2. $K = f^*(K')$ holds for a perverse sheaf K' on X' .

Proof. The direction 2. \Rightarrow 1. is obvious from $f \circ T_x = f$. For the converse first suppose that K is an irreducible perverse sheaf on X . Then $K = i_* j_{!*} L[\dim(U)]$ for a smooth etale sheaf L on U , where $j : U \hookrightarrow \overline{U}$ is an open embedding and $i : \overline{U} \rightarrow X$ is an irreducible closed subvariety \overline{U} of X . The image U' of U in X' is an irreducible subvariety of X' , and $f : U \rightarrow U'$ is an etale covering. The sheaf L defines a local

system, in other words corresponds to an irreducible finite dimensional representation $\rho : \pi_1(U, x_0) \rightarrow Gl(r, \Lambda)$. The question is now, whether this representation comes, via the natural morphism $\pi_1(U, x_0) \rightarrow \pi_1(U', x'_0)$, from an irreducible representation $\rho' : \pi_1(U, x_0) \rightarrow Gl(r, \Lambda)$. Necessarily ρ has to be invariant under the action of the covering group H of the Galois covering $f : U \rightarrow U'$, i.e. for some $T_h \in Gl(r, \Lambda)$

$$\rho(hgh^{-1}) = T_h \rho(g) T_h^{-1} \quad , \quad h \in H .$$

In fact by $T_x^*(K) \cong K$ for $x \in H$ this necessary condition is satisfied. The T_h are unique up to a scalar in Λ^* . To extend ρ to a representation ρ' amounts to show $T_{h_1 h_2} = T_{h_1} T_{h_2}$ for a suitable choice of the T_{h_i} (the obstruction for this is the Schur multiplier $H^2(H, \Lambda^*)$), and the extension ρ' is unique up to a twist by a character $\chi : H \rightarrow \Lambda^*$. If H is cyclic and generated by x , the descend condition $(T_h)^{ord(x)} = id$ can be solved in Λ^* by extracting an n -th root of unity (Λ is an algebraically closed field). Thus in the cyclic case the irreducible representation ρ can be extended to an irreducible representation ρ' in the sense above. Therefore $L \cong f^*(L')$ for some smooth etale sheaf L' on U' . Hence $K \cong f^*(K')$ for $K' = i'_* j'_{!*} L'$. Since H is a product of cyclic groups, the same holds for the abelian group H .

Now drop the assumption that K is irreducible. We still assume $T_x^*(K) \cong K$ for all $x \in H$. Then K is isomorphic to a direct sum of perverse sheaves of the form $\bigoplus_{x \in H/H_0} T_x^*(M)$ for a perverse sheaf M such that $T_x^*(M) \cong M$ for $x \in H$ holds if and only if $x \in H_0 \subset H$ for a certain subgroup H_0 of H . Again we ask, whether there exists K' on $X' = X/H$ such that $K \cong f^*(K')$ holds. By our previous argument we can descend M to X/H_0 . So without restriction of generality we may assume $H_0 = 0$. Writing $M = i_* j_{!*} L[\dim(U)]$ for a local system L , we can assume that H stabilizes $Y = \overline{U}$. Notice $Rf_*(i_* j_{!*} K|_U) = i'_* j'_{!*} (Rf_{U,*}(K|_U))$. Hence we can reduce to a suitable U (perhaps by some shrinking if necessary). Using Mackey's lemma $Rf_*(L)$ is irreducible, since $f^*(Rf_*(L)) \cong \bigoplus_{x \in H} T_x^*(L)$ (for representations of the fundamental group, opposed to the situation of representations of $\mathbf{G}(X)$, f_* corresponds to induction and f^* corresponds to restriction). Hence $L' = Rf_*(L)$ is irreducible and $f^*(L') = f^*(Rf_*(L)) \cong \bigoplus_{x \in H} T_x^*(L)$. \square

Corollary 1. *For an irreducible perverse sheaf K' on X' the pullback $K = f^*(K')$ is semisimple and the translations T_x^* , $x \in \text{Kern}(f)$ act transitively on the irreducible constituents of K .*

Proof. The maximal stable semisimple subobject S of K is stable under the isomorphisms T_x^* for $x \in \text{Kern}(f)$. Therefore by lemma 2 it is of the form $S = f^*(S')$ and then it easily follows by taking the direct image f_* that $S'_\chi \cong K'$ holds for some

character χ whose restriction to X is trivial. Hence $f^*(S'_\chi) = f^*(S') = f^*(K')$, and hence $S = K$. Replacing S by the span of all translates of a simple submodule, the same conclusion holds. \square

3 Negligible perverse sheaves

In this section we assume $k = \mathbb{C}$. By definition *negligible* objects defining the subcategory N_{Euler} of $D^{ss}(X)$ are semisimple complexes K , whose irreducible perverse constituents K_i have Euler characteristic $\chi(K_i) = 0$. The perverse constituents of a semisimple complex $K = \bigoplus_{\nu} {}^p H^\nu(K)[- \nu]$ with perverse cohomology sheaves ${}^p H^\nu(K)$ are the irreducible perverse constituents of the perverse sheaves ${}^p H^\nu(K)$. A complex in $D^{ss}(X)$ will be called *clean*, if it does not contain simple constituents from N_{Euler} . A complex K in $D^{ss}(X)$ decomposes $K = M \oplus R$, where M is a clean complex and R is negligible and M and R are unique up to isomorphism. For characters χ of the fundamental group $\pi_1(X, 0) \rightarrow \Lambda^*$ of an abelian variety X the character twist K_χ of a complex $K \in N_{Euler}$ again is in N_{Euler} . For $k = \mathbb{C}$ it is shown in [KrW], [W] that an irreducible perverse sheaf K in N_{Euler} is of the form

$$K_\chi = q^*(L)[\dim(A)]$$

for some character χ of $\pi_1(X, 0)$, some quotient $q : X \rightarrow X/A$ with respect to an abelian subvariety $A \subseteq X$ of dimension > 0 , and some irreducible perverse sheaf L on X/A . Of course we can assume that the perverse sheaf L on X/A is clean. If A is chosen in this way, the dimension $l = \dim(A)$ will be called *depth*(K). In general, for a complex $K \in N_{Euler}$, we define *depth*(K) to be the minimum of all *depth*(K_i) for the irreducible perverse constituents K_i of K . For a semisimple complex the stabilizer $Stab(K)$ of K as an abstract group is the subgroup of $X(\mathbb{C})$ defined by all $x \in X(\mathbb{C})$ for which $T_x^*(K) \cong K$ holds.

Lemma 3. *For a semisimple perverse sheaf K the stabilizer $Stab(K)$ is Zariski closed in X .*

Proof. By the next lemma the proof is easily reduced to the case where K is irreducible; so let us assume that K is irreducible. The Zariski closure Z of the group $Stab(K)$ is a closed subgroup of X , hence $A = Z^0$ is an abelian subvariety of X . Since $x \in Stab(X)$ iff $T_x^*(K) \cong K$ iff $\mathcal{H}^0(K * K^\vee)_x \neq 0$ (see [BN]), $Stab(X)$ is a constructible subgroup of X . Hence $Stab(K) \cap A$ is a constructible dense subgroup of A , and therefore contains a Zariski open subset U of A . If we replace U by

$V = U \cap -U$, then $V_1 = V$, $V_2 = V + V$, $V_3 = V + V + V$, .. etc. are dense open subsets of X contained in $Stab(K)$. Therefore $Stab(K)$ contains $W = \bigcup_{i=1}^{\infty} V_i$. Since W is a Zariski open dense subgroup of A , finitely many translates of W cover A and therefore A is the union of finitely many cosets of W . Hence W is Zariski closed, and this implies $W = A$ and $Stab(K) = Z$. \square

Lemma 4. *Suppose K is a semisimple complex and $A \subseteq Stab(K)$ is an abelian variety. Then A is contained in $Stab(K_i)$ for each irreducible perverse constituent K_i of K .*

Proof. Let K_1, \dots, K_n be a set of representatives for the isomorphism classes of the simple constituents of ${}^p H^v(K)$. So there are n isotopic blocks in ${}^p H^v(K)$. By looking at Jordan-Hölder series, for $x \in A$ we get $T_x^*(K_i) \cong K_{x(i)}$ so that $x \mapsto (x(i))$ defines a homomorphism of A to the permutation group S_n . Hence a subgroup A' of A of finite index stabilizes each K_i up to isomorphism: $A' \subseteq A \cap Stab(K_i) \subseteq A$ for $i = 1, \dots, n$. Hence $A \cap Stab(K_i)$ has finite index in A . For simple K_i furthermore $A \cap Stab(K_i)$ was shown to be closed (see the proof of lemma 3). This implies $A \cap Stab(K_i) = A$ for all i , since A is connected. Therefore $A \subseteq Stab(K_i)$ and $A' = A$. \square

By lemma 3 the connected component $A = Stab(K)^0$ of $Stab(K)$ is an abelian subvariety of X called the connected stabilizer of K . For an irreducible perverse sheaf K the dimension of the connected stabilizer is $depth(K)$.

Lemma 5. *For a complex abelian variety X let $f : Y \rightarrow X$ and $g : X \rightarrow Z$ denote isogenies of abelian varieties. For negligible semisimple objects K in $D^{ss}(X)$ the following holds:*

1. *Any character twist K_χ of K is negligible.*
2. *$f^*(K)$ is a negligible semisimple complex with $depth(f^*(K)) = depth(K)$.*
3. *$g_*(K)$ is a negligible semisimple complex with $depth(g_*(K)) = depth(K)$.*
4. *$g : Stab(K)^0 \rightarrow Stab(g_*(K))^0$ and $f : Stab(f^*(K))^0 \rightarrow Stab(K)^0$ are isogenies for a simple perverse sheaf K .*
5. *For most characters χ of $\pi_1(X, 0)$ the twist K_χ is acyclic, i.e. $H^\bullet(X, K_\chi) = 0$.*
6. *If K is perverse, there exists χ such that $H^v(X, K_\chi) \neq 0$ holds for some $v \neq 0$.*

Proof. For the proof we may assume K is perverse. For the first assertion see [KrW, cor.10]. For the second and third assertion we use that $\chi(K) = 0$ for a perverse sheaf K implies $\chi(K_i) = 0$ for all irreducible constituents K_i of K (see [FK]). For an isogeny $f : Y \rightarrow X$ with kernel F the complex $f^*(K)$ is perverse, and $Rf_*(f^*K) = \bigoplus_{\chi \in F^*} K_\chi$. Since character twists K_χ of a negligible perverse sheaf are negligible, hence $\chi(Rf_*(f^*(K))) = \chi(f^*(K)) = 0$ so that all constituents of $f^*(K)$ are negligible. Similarly $g_*(K)$ is perverse and $\chi(g_*(K)) = \chi(K) = 0$. This proves the first part of the assertions 2 and 3.

The remaining part of the assertions 2 and 3 on the depth follows from the special case, where K is an irreducible perverse sheaf. So let us assume K is irreducible perverse. For the abelian variety $A = \text{Stab}(K)^0$ and $y \in f^{-1}(A)$ then $\varphi : K \cong T_{f(y)}^*(K)$ implies $f^*(\varphi) : f^*(K) \cong f^*(T_{f(y)}^*(K)) = T_y^*(f^*(K))$, thus $y \in \text{Stab}(f^*(K))$ and the abelian variety $f^{-1}(A)^0$ stabilizes $f^*(K)$ and by lemma 4 then also each irreducible component of $f^*(K)$. This implies $\text{depth}(f^*(K)) \geq \text{depth}(K)$. Conversely, if an abelian variety $B \subseteq Y$ stabilizes a constituent of $f^*(K)$, then for each $y \in B$ there exists a nontrivial map in

$$\text{Hom}(f^*(K), T_y^*(f^*(K))) = \text{Hom}(f^*(K), f^*(T_{f(y)}^*(K))) = \text{Hom}(K, \bigoplus_{\chi \in F^*} T_{f(y)}^*(K_\chi)) ,$$

and hence $T_{-f(y)}^*(K) \cong K_\chi$ holds for some $\chi \in F^*$, since K is irreducible. Thus for a subgroup $U \subseteq B$ of finite index and $y \in U$ we get $f(y) \in \text{Stab}(K)$. But $f(U) \subseteq \text{Stab}(K) \cap f(B) \subseteq f(B)$ shows that $\text{Stab}(K) \cap f(B)$ is of finite index in $f(B)$. Hence $\text{Stab}(K)$ contains $f(B)$, and $\text{depth}(f^*(K)) \leq \text{depth}(K)$. This shows $\text{depth}(f^*(K)) = \text{depth}(K)$, and $f : \text{Stab}(f^*(K))^0 \rightarrow \text{Stab}(K)^0$ is an isogeny.

For the assertion on $\text{depth}(g_*(K))$ again let K be a simple perverse sheaf. Put $A = \text{Stab}(K)^0 \subset X$, then $g(A)$ stabilizes $g_*(K)$ and therefore all its constituents. This gives $\text{depth}(K) \leq \text{depth}(g_*(K))$. Conversely, if L is an irreducible constituent of $g_*(K)$ and $\varphi : T_z^*(L) \cong L$ an isomorphism for $z \in \text{Stab}(L)$, we choose $x \in X$ such that $g(x) = z$. By the adjunction formula for perverse sheaves $\text{Hom}(L, g_*(K)) = \text{Hom}(g^*(L), K)$ there exists a nontrivial morphism $g^*(L) \rightarrow K$. Hence K is an irreducible constituent of $g^*(L)$. Since $g^*(\varphi) : g^*(T_z^*(L)) \cong g^*(L)$ and $g^*(T_z^*(L)) = T_x^*(g^*(L))$ implies $g^*(L) \cong T_x^*(g^*(L))$, we get $T_x^*(K) \cong K$ for all $x \in g^{-1}(\text{Stab}(L))^0$ by lemma 4. Hence $g^{-1}(\text{Stab}(L))^0 \subset \text{Stab}(K)^0$, and this proves $\text{depth}(g_*(K)) \leq \text{depth}(K)$. Therefore $\text{depth}(g_*(K)) = \text{depth}(K)$.

For assertion 5 again assume K is irreducible. Then for the connected stabilizer $A \subseteq X$ of K there exists an isogeny $f : A \times B \rightarrow X$ such that $H^\bullet(X, K) \subseteq H^\bullet(A \times B, L_\psi \boxtimes K_B) = H^\bullet(A, L_\psi) \otimes_\Lambda H^\bullet(B, K_B)$. For a twist of K with a character χ , whose pullback (χ_1, χ_2) under $\pi_1(f) : \pi_1(A, 0) \times \pi_1(B, 0) \rightarrow \pi_1(X, 0)$ satisfies

$\chi_1 \neq \psi^{-1}$, the cohomology of K_χ vanishes. The same argument shows for $\chi_1 = \psi^{-1}$ that the cohomology $H^\bullet(A, \Lambda[\dim(A)]) \otimes_\Lambda H^\bullet(B, K_{B, \chi_2})$ does not vanish in degree $v = \dim(A)$ for most χ_2 (by the main result of [KrW]). Since any pair of characters (χ_1, χ_2) can be written as the restriction of some character χ of $\pi_1(X, 0)$, this implies $H^{\dim(A)}(X, K_{\chi\chi_0}) \neq 0$ for some of the finitely many characters χ_0 of $\pi_1(X, 0)$ which are trivial on the image of $\pi_1(f) : \pi_1(A, 0) \times \pi_1(B, 0) \rightarrow \pi_1(X, 0)$. This proves assertion 6. \square

Lemma 6. *For a semisimple complex T on X and an abelian subvariety $A \subseteq X$ let $f : X \rightarrow B = X/A$ be the quotient map. Suppose $\text{depth}(T) = \dim(B) > 0$. Then for most χ the complex $Rf_*(T_\chi)$ is negligible of depth $= \dim(B)$ or zero. If in addition $\text{Stab}(T)^0$ maps surjectively onto B , then there exists a character χ so that the complex $Rf_*(T_\chi)$ is nonzero; if finally also $X = A \times B$ holds, then there exist a character χ_0 so that for most characters χ of $\pi_1(A, 0)$, extended to characters of $\pi_1(X, 0)$ that are trivial on $\pi_1(B, 0)$, the complex $Rf_*(T_{\chi_0\chi})$ is negligible of depth $\dim(B)$ but not acyclic.*

Proof. For the proof we may replace T by one of its semisimple perverse sheaves $K = {}^p H^v(T)$ on X , without restriction of generality. For $D = \text{Stab}(K)^0 \cap A$ by our assumptions $\dim(D) = 0$ holds iff $\text{Stab}(K)^0$ surjects onto B .

First suppose $\dim(D) > 0$. Then there exists an isogeny $g : D \times Y \rightarrow X$ so that $g^*(K) = L_\psi \boxtimes K_Y$ for some perverse sheaf K_Y on Y (see [W]). K is a summand of $g_* g^*(K)$. Hence, as in the proof of lemma 5.5, we can replace X by $D \times Y$ and f by $f \circ g$ to show $Rf_*(K_\chi) = 0$ for most χ . Indeed, then it is clear that $Rf_*(K_\chi) = 0$ holds whenever the restriction of χ to $\pi_1(D, 0)$ is different from ψ^{-1} , since $f \circ g$ factorizes over the projection $D \times Y \rightarrow Y$.

Now assume $\dim(D) = 0$. Then B stabilizes $Rf_*(K_\chi)$, which therefore is negligible for all $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$ or is zero. By our assumptions there exists an isogeny $g : A \times \text{Stab}(K)^0 \rightarrow X$ defined by $g(x, a) = a + x$ so that $f \circ g(x, a) = f(x)$. Choose an isogeny $\tilde{g} : X \rightarrow A \times \text{Stab}(K)^0$ so that $\tilde{g} \circ g$ is multiplication by an integer n

$$\begin{array}{ccccc} A \times \text{Stab}(K)^0 & \xrightarrow{g} & X & \xrightarrow{\tilde{g}} & A \times \text{Stab}(K)^0 \\ \downarrow pr_2 & & \downarrow f & & \downarrow pr_2 \\ \text{Stab}(K)^0 & \xrightarrow{g} & X/A & \xrightarrow{\tilde{g}} & \text{Stab}(K)^0 \end{array}$$

By the arguments of [W] used for the proof of theorem 2 of loc. cit. it follows that $g^*(K) \cong K_A \boxtimes L_\psi$ holds for some invariant irreducible perverse sheaf L_ψ on the

factor $Stab(K)^0$ and some perverse sheaf K_A on the factor A . By our assumptions on the depth K_A is not negligible for at least for one perverse cohomology degree v (fixed above). Notice $g_*g^*(K) = g_*(K_A \boxtimes L_\psi)$ is a direct sum of twists K_χ of K including K itself. To show $Rf_*(K_\chi) \neq 0$ for one χ , we may therefore replace f by pr_2 , X by $A \times Stab(K)^0$ and K by any irreducible clean constituent L of $\tilde{g}_*(K)$. This immediately follows by the commutative diagram above. Now $\tilde{g}_*(K)$ is a retract of $\tilde{g}_*g_*g^*(K) = n_*K_A \boxtimes n_*L_\psi$, hence isomorphic to a direct sum of sheaves $n_*K_A \boxtimes L_{\psi'}$. Notice n_*K_A is not negligible and therefore can be replaced by its nonvanishing clean part. So without restriction of generality we are thus in the situation, where $X = A \times B$ and $K = K_A \boxtimes L_{\psi'}$ holds for a clean perverse sheaf $K_A \neq 0$. But then $Rpr_{2,*}(K_{\chi_A \boxtimes \chi_B}) = H^\bullet(A, K_{A, \chi_A}) \otimes L_{\psi' \chi_B}$. So for $\chi_A \chi_0 = (\chi_A, 1)$ our claim follows, since $\chi(H^\bullet(A, K_{A, \chi_A})) = \chi(K_{A, \chi_A}) = \chi(K_A) \neq 0$ holds for all characters $\chi_A : \pi_1(A, 0) \rightarrow \Lambda^*$ by [KrW, cor.10]. \square

4 Grothendieck rings

Assuming $k = \mathbb{C}$ for a *simple complex* abelian variety of dimension g let $K^0(P(X))$ denote the Grothendieck group of the category $P(X)$ of semisimple perverse sheaves on X , and let $K_*^0(X)$ be the tensor product of $K^0(P(X))$ with the polynomial ring $\mathbb{Z}[x, x^{-1}]$. We define a $\mathbb{Z}[x, x^{-1}]$ -linear homomorphism

$$h : K_*^0(X) \rightarrow \mathbb{Z}[x, x^{-1}]$$

on generators by

$$h(K) = \sum_{v \in \mathbb{Z}} \dim_\Lambda(H^v(X, K)) \cdot x^v$$

for perverse sheaves K . The Grothendieck ring $\bar{K}_*(X)$ of the tensor category $\bar{P}(X)$, tensored with the polynomial ring $\mathbb{Z}[x, x^{-1}]$, is obtained from $K_*^0(X)$ as the quotient by the classes of negligible perverse sheaves. Since X is simple [KrW] shows that the simple negligible perverse sheaves are the character twists $L_\chi[g]$ of the constant perverse sheaf $\delta_X = \Lambda_X[g]$, and $h(L_\chi[g]) = 0$ holds for nontrivial χ . The composite homomorphism h_*

$$K_*^0(X) \rightarrow \mathbb{Z}[x, x^{-1}] \rightarrow \mathbb{Z}[x, x^{-1}]/(x + x^{-1} + 2)^g$$

is h composed with the quotient map modulo the ideal generated by $h(\delta_X)$. Notice $\mathbb{Z}[x, x^{-1}]/h(\delta_X) = \mathbb{Z}[x, x^{-1}]/(x + x^{-1} + 2)^g \cong \mathbb{Z}[x, x^{-1}]/(x^2 + 1 + 2x)^g \cong \mathbb{Z}[y, (1 -$

$y)^{-1}]/y^{2g} = \mathbb{Z}[y]/y^{2g}$ by the substitution $y = 1 + x$, since $-x = 1 - y$ is inverse to $1 + y + \dots + y^{2g-1}$ in $\mathbb{Z}[y]/y^{2g}$.

Now if $X = \prod_{j=1}^r A_j$ is a *product of simple complex* abelian varieties A_j of dimensions g_j , we consider the projection $f_i : X_i = \prod_{j=1}^i A_j \rightarrow X_{i-1} = \prod_{j=1}^{i-1} A_j$. Similar to above define

$$K_*^0(X_i) = K_0(\text{Perv}(X_i)) \otimes \mathbb{Z}[x_1, x_1^{-1}, \dots, x_i, x_i^{-1}]$$

and the ring

$$R = \bigotimes_{i=1}^r \mathbb{Z}[x_i, x_i^{-1}] / (x_i + x_i^{-1} + 2)^{g_i} \cong \mathbb{Z}[y_1, \dots, y_r] / (y_1^{2g_1}, \dots, y_r^{2g_r}) .$$

$R_{\mathbb{Q}} = R \otimes \mathbb{Q}$ is a local Artin ring. Inductively define $h = h_r : K_0^*(X) \rightarrow \mathbb{Z}[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$ and more generally

$$h_i : K_0^*(X_i) \rightarrow \mathbb{Z}[x_1, x_1^{-1}, \dots, x_i, x_i^{-1}] \quad , \quad i = 1, \dots, r$$

such that $h_i = h_{i-1} \circ p_i$ for $p_i : K_0^*(X_i) \rightarrow K_0^*(X_{i-1}) \otimes \mathbb{Z}[x_i, x_i^{-1}]$ as a $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_i, x_i^{-1}]$ -linear extension of the map defined for perverse sheaves $K \in \text{Perv}(X_i)$ by

$$p_i(K) = \sum_{v \in \mathbb{Z}} {}^p H^v(Rf_{i*}(K)) \cdot x_i^v .$$

The p_i and therefore the h_i are *ring homomorphisms*. For simple objects K and L this follows from the identity ${}^p H^k(Rf_{i*}(K * L)) \cong \bigoplus_{v+\mu=k} {}^p H^v(Rf_{i*}(K)) * {}^p H^\mu(Rf_{i*}(L))$ in $\bar{P}(X)$, which follows from the decomposition theorem. Define

$$h_* : K_*^0(X) \longrightarrow R ,$$

as the composite of h and the projection to R .

Lemma 7. *If a complex abelian variety X is a product $X = \prod_{j=1}^r A_j$ of simple abelian varieties A_j , the ring morphism h_* factorizes over the quotient*

$$K_*^0(X) / (h(\delta_{X_1}), \dots, h(\delta_{X_r}))$$

of $K_*^0(X)$ and induces a ring homomorphism

$$\bar{h}_* : \bar{K}_*(X) \longrightarrow R .$$

Proof. We have to show $h_*(K) = 0$ for all simple negligible perverse sheaves K on X . By [W] any negligible perverse sheaf in $P(X)$ is a direct sum of simple perverse sheaves K , each of which is A -invariant for a certain simple abelian variety $A \subset X$ of dimension $\dim(A) > 0$ in the sense that $T_x^*(K) \cong K$ holds for all closed points x in A . By lemma 4 all irreducible simple perverse components of ${}^p H^i(Rf_{r*}(K))$ are $f_r(A)$ -invariant. If $f_r(A)$ is not zero, the complex $Rf_{r*}(K)$ is negligible and our claim follows by induction on r in this case. If $f_r(A)$ is zero, then $A = A_r$ and $K \cong K_{r-1} \boxtimes L_{\chi_r}$ for some perverse sheaf K_{r-1} on X_{r-1} and some twisting perverse character sheaf L_{χ_r} on A_r attached to a one dimensional character of the fundamental group $\pi(A_r, 0)$. Then $Rf_{r*}(K) = H^\bullet(A_r, L_{\chi_r}) \otimes_{\Lambda} K_{r-1}$. Hence the image $h_*(K)$ of $h(K)$ is zero in R , since $p_r(K) = (x_r + x_r^{-1} + 2)^{g_r} \cdot K_{r-1}$ or $pr_r(K)$ is zero. \square

5 Finite component groups H

The group $\pi_0(G(X))$ of connected components of $G(X)$ is defined as the projective limit of the finite groups $\pi_0(G)$ of Zariski connected components, where the limit is taken over all algebraic quotient groups G of $G(X)$. Notice that $\pi_0(G(X))$ is an abelian group, if all these finite component groups $\pi_0(G)$ are abelian, and the converse is also true.

The representation categories of the finite quotient groups H of G or $G(X)$ are full tensor subcategories of the representation category of $G(X)$. For complex abelian varieties the super-Tannakian category $sRep_{\Lambda}(G(X), \mu)$ is equivalent to an ordinary neutral Tannakian category $Rep_{\Lambda}(G(X))$ by [KrW]; furthermore for finite abelian groups H this subcategory attached to H as an additive category is generated by skyscraper sheaves supported in torsion points of $X(\mathbb{C})$. Hence theorem 2 follows from

Theorem 3. *For a complex abelian variety X the group $\pi_0(G)$ of connected components of an algebraic quotient group G of $G(X)$ is abelian.*

Corollary 2. *For a complex abelian variety X the group $\pi_0(G(X))$ of connected components of $G(X)$ with respect to the Zariski topology is abelian.*

By Tannakian arguments to prove theorem 3 it suffices that any full tensor subcategory in $Rep_{\Lambda}(G(X))$ isomorphic to $Rep_{\Lambda}(H)$, where H is a finite group, is generated as an additive category by (perverse) skyscraper sheaves δ_x , where the points x are contained in a finite torsion subgroup $H \subseteq X(\mathbb{C})$.

The regular representation of H . Over an algebraically closed field Λ of characteristic zero the category $\text{Rep}_\Lambda(H)$ of representations of a finite group H contains the regular representation \bar{K} of H defined by left multiplication of H on the group ring

$$\bar{K} = \Lambda[H] .$$

\bar{K} is self dual in the sense that $\bar{K} \cong \bar{K}^\vee$ holds and its dimension is the cardinality d of H . Notice

$$\Lambda[H] \otimes_\Lambda \Lambda[H] \cong d \cdot \Lambda[H]$$

and by Wedderburn's theorem $\Lambda[H]$ contains all irreducible representations of H up to isomorphism

$$\bar{K} \cong \bigoplus_{i=1}^h d_i \cdot \bar{K}_i \quad , \quad \dim_\Lambda(\bar{K}_i) = d_i$$

for a set of representatives \bar{K}_i of the isomorphism classes of the irreducible representations of H over Λ .

Suppose $\text{Rep}_\Lambda(H)$ as a tensor category is equivalent to a tensor subcategory \mathcal{T}_H of the Tannaka category $\bar{P}(X) = \text{Rep}_\Lambda(G(X))$, where $\bar{P}(X)$ is the category of semisimple perverse sheaves on X up to negligible perverse sheaves. Then every representation of H can be considered as an object of $\bar{P}(X) = \text{Rep}_\Lambda(G(X))$. So the regular representation of H is represented by a semisimple perverse sheaf denoted K . This perverse sheaf $K \in \text{Perv}(X, \Lambda)$ is uniquely determined by \bar{K} if it is chosen to be a clean perverse sheaf. Then K is a self dual in the sense $K^\vee \cong K$ for $K^\vee = (-id_X)^*(D(X))$ for the Verdier dual $D(K)$ of K , and

$$K * K \oplus T' = d \cdot K \oplus T \quad , \quad \chi(K) = \dim(K) = d$$

holds for some finite direct sums T, T' of complex shifts of negligible perverse sheaves on X . It is easy to see that $T' = 0$ (see section 6). The perverse constituents $K_i \in P(X)$ representing the irreducible constituents $\bar{K}_i \in \bar{P}(X)$ up to isomorphism are uniquely defined by the \bar{K}_i , and

$$K = \bigoplus_{i=1}^h d_i \cdot K_i \quad , \quad \chi(K_i) = d_i > 0$$

is a semisimple perverse sheaf on X . A semisimple perverse sheaf obtained in this way will be called an *H-regular* perverse sheaf on X . Character twists K_χ of *H-regular* perverse sheaves K are *H-regular* perverse sheaves, since twisting with a character sheaf L_χ defines a tensor functor (see [KrW]).

6 Strongly d -regular perverse sheaves

Definition. Let d be an integer ≥ 1 . A semisimple complex K on X is called *weakly d -regular*, if $K^\vee \cong K$ and

$$K * K \oplus T' \cong d \cdot K \oplus T$$

hold for certain direct sums T and T' of complex shifts of negligible perverse sheaves on X . K is called *d -regular* if $T' = 0$, and *strongly d -regular* if $T' = 0$ and $T = 0$ and K is a perverse sheaf. A H -regular and strongly d -regular perverse sheaf is called *strongly H -regular*.

Example. For H -regular perverse sheaves K the perverse sheaf $m \cdot K$ is weakly d -regular for $d = m \cdot \#H$.

If K is weakly d -regular on X , the subcategory whose objects are finite direct sums of the images \overline{K}_i of the simple constituents K_i of K is a tensor subcategory of $\overline{P}(X)$, hence defines a finite quotient group H of $G(X)$.

Lemma 8. Let K be a weakly d -regular complex on a complex abelian variety and $K = M \oplus R$ its decomposition into a clean complex M and a negligible complex R . Then the complex M is a perverse sheaf.

Proof. For $K = \bigoplus_v M_v[v]$ with $M_v = {}^p H^v(K)$ let v be maximal (or minimal) with $M_v \notin N_{Euler}$. The representation of $\mathbf{G}(X) = \mathbb{G}_m \times G(X)$ associated to M_v is an isotopic multiple of the character $t \mapsto t^v$ if restricted to \mathbb{G}_m . Therefore the tensor product $K * K$, restricted to \mathbb{G}_m , contains the character $t \mapsto t^{2v}$. This contradicts the maximality of v unless $v = 0$ and proves, that K is a sum of a perverse sheaf and a negligible complex. \square

Lemma 9. Any weakly d -regular complex K has the form

$$K = M \oplus R$$

for a d -regular clean perverse sheaf M and a negligible complex R .

Proof. By lemma 8 the semisimple complex K is of the form $K = M \oplus R$ for a negligible perverse sheaf R and a perverse sheaf M without negligible summands. Since $d \cdot M$ is clean and

$$M^{*2} \oplus (2 \cdot M * R \oplus R^{*2} \oplus T') \cong d \cdot M \oplus (d \cdot R \oplus T),$$

the summand $(2 \cdot M * R \oplus R^{*2} \oplus T')$ has to be a summand of $d \cdot R \oplus T$. Hence $M^\vee \cong M$ and $M^{*2} \cong M \oplus T''$ for some negligible complex T'' . \square

Lemma 10. *For an isogeny $g : X \rightarrow Y$ of complex abelian varieties the direct image of a d -regular perverse sheaf K on X is a d -regular perverse sheaf on Y . Furthermore, twists K_χ of d -regular perverse sheaves K are d -regular.*

Proof. By the decomposition theorem $g_*(K)$ is semisimple. The other properties follow from the fact that g_* is a tensor functor. Notice $g_*(T')$ and $g_*(T)$ remain negligible by lemma 5.5. \square

Now consider the following

Assertion $Tor(n)$. *For all complex abelian varieties X of dimension $\dim(X) \leq n$ any clean d -regular complex K on X is a perverse skyscraper sheaf concentrated in torsion points of $X(\mathbb{C})$.*

Assertion $Reg(n)$. *For all complex abelian varieties X of dimension $\dim(X) \leq n$ any clean d -regular complex K on X is strongly d -regular*

$$\boxed{K^{*2} \cong d \cdot K \quad , \quad K \cong K^\vee}.$$

Obviously $Tor(n)$ implies $Reg(n)$.

Proposition 1. *$Tor(m)$ for all $m < n$ implies $Reg(n)$.*

Proof. By lemma 9 a clean d -regular complex K is a perverse sheaf. So for $K * K \cong d \cdot K \oplus T$ we have to show $T = 0$.

First step. Suppose A and B are abelian subvarieties of X with finite intersection. Then there exists an isogeny $g : X \rightarrow Z = A \times B$. Assume K is a d -regular clean perverse sheaf such that $K * K \cong d \cdot K \oplus T$ with a negligible complex T . To show $T = 0$, without restriction of generality, we can replace X by $A \times B$ and K by $g_*(K)$. Indeed $M = g_*(K)$ is a d -regular perverse sheaf with $M * M \cong d \cdot M \oplus g_*(T)$ for negligible $g_*(T)$ by lemma 10; furthermore $T = 0$ iff $g_*(T) = 0$. Notice that M is clean by lemma 5.

Also notice: $Stab(T) = X \Rightarrow Stab(g_*(T)) = Z$. and the perverse adjunction formula $Hom(N, g_*(M)) = Hom(g^*(N), M)$.

Second step. For a d -regular clean perverse sheaf $K \cong K^\vee$ on X with $K^{*2} = dK \oplus T$ for a negligible complex T , we claim $depth(T) = \dim(X)$.

Suppose $depth(T) < \dim(X)$ so that there exists a constituent $N \neq 0$ of T with $depth(N) = depth(T)$. Since T is negligible $\dim(Stab(N)^0) > 0$ and by assumption

$Stab(N)^0 \neq X$. Hence there exists an abelian variety $A \subset X$ for which the natural map $A \times Stab(N)^0 \rightarrow X$ is an isogeny. The quotient map $q : X \rightarrow B = X/A$ defines an isogeny $q : Stab(N)^0 \rightarrow B$. For the proof of the claim we use step 1 above and lemma 5.3 to reduce to the special case where $X = A \times Stab(T)^0$ and q is the projection onto the second factor. Then there exists a character χ_0 by lemma 6, so that for most characters χ of $\pi_1(A, 0)$ the complex

$$S(\chi) = Rq_*(T_{\chi_0\chi})$$

is not acyclic but negligible of depth $\dim(B)$ on B . $S(\chi)$ is a direct sum of complex shifts of perverse sheaves invariant under B , and since it is not acyclic some shift $\delta_B[v]$ of $\delta_B = \Lambda_B[\dim(B)]$ occurs as a constituent of $S := S(\chi)$; of course v may depend on χ .

The complex $M = Rq_*(K_{\chi_0\chi})$ is d -regular (lemma 10) and

$$M * M \cong d \cdot M \oplus Rq_*(T_{\chi_0\chi}) = d \cdot M \oplus S.$$

For any χ the complex M decomposes by lemma 9 into a direct sum

$$M = M_{clean} \oplus R$$

for a negligible complex R and a clean perverse sheaf M_{clean} , depending on χ . Since $\dim(B) < \dim(X)$ the induction assumption $Reg(m)$ implies $M_{clean}^{*2} = d \cdot M_{clean}$. Hence

$$(2 \cdot M_{clean} * R) \oplus R^2 = d \cdot R \oplus S.$$

Furthermore, by the induction assumption $Tor(m)$ the perverse sheaf M_{clean} is a skyscraper sheaf of dimension $\chi(M_{clean}) = d$. Hence $2 \cdot M_{clean} * R$ is a direct sum of $2d$ translates of R and for $R \neq 0$ the left side can not be completely contained in the term $d \cdot R$ on the right side. Hence for $R \neq 0$ some translate of each simple constituent of R occurs as constituent of S . For $R \neq 0$ this implies $Stab(R) = B$. Therefore up to complex shifts all simple constituents of R are of the form L_ψ for characters $\psi : \pi_1(B, 0) \rightarrow \Lambda^*$.

We claim that R is not acyclic, since otherwise δ_B is not a constituent of R . Since $L_\psi * P = H^\bullet(B, P_\psi) \cdot L_\psi$ holds for all P , then $(2 \cdot M_{clean} * R) \oplus R^2$ and hence $d \cdot R \oplus S$ would also be acyclic contradicting that $S = S(\chi)$ is not acyclic. Using the hard Lefschetz theorem this shows, that there exists some index $v_0 = v_0(\chi) \geq 0$ so that the perverse sheaf M decomposes in the form

$$M = M_{clean} \oplus \delta_B[-v_0] \oplus \dots$$

This implies $\bigoplus_{i \geq \dim(B)} H^i(B, M) \neq 0$. Consider the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{q} & B \\ p \downarrow & & \downarrow \tilde{p} \\ A & \xrightarrow{\tilde{q}} & \text{Spec}(k) \end{array}$$

By the perverse Leray spectral sequence (which degenerates by the decomposition theorem) then

$$\bigoplus_{v+\mu \geq \dim(B)} H^v(A, {}^p R^\mu p_*(K_{\chi_0 \chi})) \cong \bigoplus_{i \geq \dim(B)} H^i(A \times B, K_{\chi_0 \chi}) \cong \bigoplus_{i \geq \dim(B)} H^i(B, M) \neq 0$$

for most characters χ of $\pi_1(A, 0)$. However $H^v(A, {}^p R^\mu p_*(K_{\chi_0 \chi})) = H^v(A, {}^p R^\mu p_*(K_{\chi_0})_\chi)$ vanishes for all μ , all $v \neq 0$ and most χ by the vanishing theorem of [KrW]. Hence for most χ

$$\bigoplus_{i \geq \dim(B)} H^0(A, {}^p R^i p_*(K_{\chi_0 \chi})) \neq 0$$

implies ${}^p R^i p_*(K_{\chi_0 \chi}) \neq 0$ for some $i \geq \dim(B)$. Since p is smooth of relative dimension $\dim(B)$ and $K_{\chi \chi_0}$ is perverse, [BBD], 4.24 or [KW], thm.III.11.3 then shows $i = \dim(B)$ and furthermore implies that $K_{\chi \chi_0}$ contains a B -invariant constituent. However this contradicts our assumption that K is clean. So this contradiction proves our claim

$$\text{depth}(T) = \dim(X) .$$

Third step. So now assume, K is a d -regular clean perverse sheaf satisfying $K * K \cong d \cdot K \oplus T$ with a negligible complex T . Then by step 2) the complex T is invariant by X . We have to show $T = 0$. By step 1 we can assume that $X = \prod_{i=1}^r A_i$ is a product of simple abelian varieties. For $r > 1$ we can project $q : X \rightarrow B = A_1$ with kernel $A = \prod_{i=2}^r A_i$. Then $L = Rq_*(K) = M \oplus R$ for a clean perverse sheaf M (lemma 9) and a negligible complex R on B , satisfying

$$M^{*2} \oplus (2 \cdot M * R \oplus R^{*2}) = d \cdot M \oplus (d \cdot R \oplus Rq_*(T)) .$$

The sheaf complexes in brackets are negligible complexes. So $M^{*2} \cong dM$ follows from the induction assumption $\text{Reg}(m)$. We conclude

$$(2 \cdot M * R) \oplus R^{*2} = d \cdot R \oplus Rq_*(T) .$$

Now $\text{Stab}(Rq_*(T)) = B$, since $\text{Stab}(T) = X$. Since B is simple and R is negligible, also $\text{Stab}(R) = B$. Thus $R = \bigoplus_\chi R_\chi$ and $Rq_*(T) = \bigoplus_\chi S_\chi$ where R_χ and S_χ are sums

of complex shifts of the rank one negligible perverse sheaf $L_\chi = \Lambda_B[\dim(B)]_\chi$, for characters $\chi : \pi_1(B, 0) \rightarrow \Lambda^*$. Since $L_\chi * L_{\chi'} = 0$ for $\chi \neq \chi'$ and $L_\chi^{*2} = H_B^\bullet \otimes_\Lambda L_\chi$ for $H_B^\bullet = H^\bullet(B, \Lambda)[\dim(B)]$, this implies

$$(2 \cdot M * R_\chi) \oplus R_\chi^{*2} = d \cdot R_\chi \oplus S_\chi ,$$

for each χ , where $M * R_\chi := H^\bullet(B, M_\chi) \otimes_\Lambda L_\chi$, $R_\chi := V_\chi \otimes_\Lambda L_\chi$ and $R_\chi^{*2} = (V_\chi^{\otimes 2} \otimes_\Lambda H_B^\bullet) \otimes_\Lambda L_\chi$ and $S_\chi := W_\chi \otimes_\Lambda H_A^\bullet \otimes_\Lambda L_\chi$ for $H_A^\bullet = H^\bullet(A, \Lambda)[\dim(A)]$ and some W_χ . Without restriction of generality we may now suppress χ . Let denote $P_U(x) \in \mathbb{Z}[x, x^{-1}]$ the Poincare polynomial of a graded Λ -vector spaces $U = U^\bullet$. With this notation

$$P_V(x) \cdot \left(2P_M(x) - d + P_V(x) \cdot (2 + x + x^{-1})^{\dim(B)} \right) = P_W(x) \cdot \left(2 + x + x^{-1} \right)^{\dim(A)} .$$

Furthermore $1 - \dim(A) \leq \deg_x(P_V) \leq \dim(A) - 1$, since M is a perverse sheaf on B without A -invariant (and hence negligible) constituents. Clearing the denominators, which are powers of x , we obtain a relation in the polynomial ring $\mathbb{Q}[x]$. We compare the factorization of the prime polynomial $(x + 1)$ on both sides. The right side is divisible by $(x + 1)^{2\dim(A)}$. The polynomial, arising from $2P_M(x) - d$ by clearing denominators, is not divisible by $(x + 1)$ at all, since $2P_M(-1) - d = d \neq 0$ follows from following computation of Euler characteristics: $P_M(-1) = \chi(M) = \chi(Rq_*(K)) = \chi(K) = d$. Thus $P_V(x)$ is divisible by $(x + 1)^{2\dim(A)}$ after clearing denominators. However this is impossible, since this polynomial in $\mathbb{Q}[x]$ has degree at most $2\dim(A) - 2$. This is a contradiction unless $P_V(x) = 0$, which means $R = 0$. But $R = 0$ implies that M is a clean d -regular perverse sheaf on B . Hence by the induction assumption $Reg(m)$ we get $S = 0$. Hence $Rq_*(T) = 0$, and since $Stab(T) = X$, this also implies $T = 0$.

Last step. It remains to consider the case where X is a simple abelian variety of dimension g , K a d -regular clean perverse sheaf on X ; in particular $K * K \cong d \cdot K \oplus T$ for negligible T . We then claim

$$H^v(X, K_\chi) = 0 \quad , \quad \forall \chi \quad \forall v \neq 0 .$$

Since K_χ has the same properties as K , we may assume $\chi = 1$. Using the ring homomorphism

$$h : K_*^0(X) \rightarrow \mathbb{Z}[x, x^{-1}] ,$$

of section 4, our claim amounts to show that the image $h(K)$ is a constant Laurent polynomial. To show this we may extend coefficients from \mathbb{Z} to \mathbb{Q} to consider

$h : K_*^0(X) \otimes \mathbb{Q} \rightarrow \mathbb{Q}[x, x^{-1}]$ instead. A necessary condition is, that the image of K under

$$h_* : K_*^0(X) \otimes \mathbb{Q} \rightarrow R_{\mathbb{Q}} = \mathbb{Q}[x, x^{-1}]/(x + x^{-1} + 2)^g$$

becomes constant. By lemma 3 this ring homomorphism h_* kills the classes of negligible complexes, hence $P = h_*(K)$ in $R_{\mathbb{Q}}$ satisfies $P^2 = dP$. Thus, either P is in the nil radical of the local Artin ring $R_{\mathbb{Q}}$, or P is a unit. In the first case $P = 0$ in $R_{\mathbb{Q}}$, since $P^2 = dP$ and hence $d^{2g-1}P = P^{2g} = 0$ in $R_{\mathbb{Q}}$. In the second case P is invertible in $R_{\mathbb{Q}}$ and hence $P = d$ in $R_{\mathbb{Q}}$.

This being said it remains to show that $h(K)$ is either zero or equal to the constant d in the Laurent ring $\mathbb{Q}[x, x^{-1}]$ itself. Here we use that K is semisimple perverse sheaf without constant, hence negligible constituents. Therefore

$$h(K) = \sum_{v=-g+1}^{g-1} {}^p H^v(X, K) \cdot x^v \in \mathbb{Q}[x, x^{-1}] ,$$

since otherwise there exist constituents in K stabilized by X by [BBD], 4.2.4 or [KW], theorem III.11.3. So suppose, as in our situation above, there exists a constant $a \in \mathbb{Q}$ and a Laurent polynomial f in x , in our case either $a = 0$ or $a = d$ in \mathbb{Q} , such that

$$h(K) - a = f(x, x^{-1}) \cdot (x + 1)^{2g} .$$

If $h(K) \neq a$ were not constant, then $h(K) - a = x^{-j} \cdot h(x)$ for a polynomial $h(x)$ of degree $\leq 2g - 2$ such that $h(0) \neq 0$. Then also $f(x, x^{-1}) = x^{-i} \cdot g(x)$ for a polynomial $g(x) \in \mathbb{Q}[x]$ such that $g(0) \neq 0$. Hence $h(x) = x^{j-i}(x+1)^{2g}g(x)$ and $x^{i-j}h(x) = (x+1)^{2g}g(x)$ imply $i = j$ putting $x = 0$. Therefore $h(x) = (x+1)^{2g}g(x)$. This gives a contradiction by a comparison of degrees. Thus $h(K) = a$ must be constant and our claim follows

$$H^v(X, K) = 0 \quad , \quad \text{for } v \neq 0 .$$

This applies for all character twists K_{χ} of K . By the Künneth formula this property is then inherited to all convolution powers K^{*r} of K and their constituents. Then $K^{*2} = d \cdot K \oplus T$ implies the same for T and all its character twist. This proves $H^v(X, T_{\chi}) = 0$ for all χ and all $v \neq 0$. This contradicts lemma 5.6 unless $T = 0$. \square

7 Perfect groups H

Let X be a complex abelian variety. Let $K \in \text{Perv}(X, \mathbb{C})$ be a d -regular semisimple perverse sheaf on X . The Tannakian category \mathcal{T} generated by K defines a finite

Tannaka group G of order dividing d . The irreducible representations of G correspond to the simple constituents of K , and K represents (a multiple of) the regular representation of G . Hence as a representation K is stable under tensor products with characters of $G^{ab} = G/[G, G]$.

The additive subcategory of \mathcal{T} generated by skyscraper sheaves δ_x in \mathcal{T} for closed torsion points $x \in X$ is stable under convolution, and hence defines a neutral Tannakian subcategory of \mathcal{T} . Its Tannaka group H can be identified with the group G^{ab} . Indeed the invertible objects $\delta_x \in \mathcal{T}$ define characters of G . Conversely by [KrW, prop. 21] any one dimensional representation of G corresponds to a skyscraper sheaf δ_x in \mathcal{T} for some torsion point x in $X(\mathbb{C})$; this and $\delta_x * \delta_y = \delta_{x+y}$ allows to identify H with a finite subgroup of a subgroup of the group of torsion points in X . Since the regular representation K is stable under convolution with all characters of G^{ab} , the formula $\delta_x * K = T_{-x}^*(K)$ implies the direction \Leftarrow of the next claim:

$$T_x^*(K) \cong K \iff x \in H .$$

For the converse direction \Rightarrow notice that $T_x^*(K) \cong K$ implies $\mathcal{H}^0(K * K^\vee)_x \neq 0$ (see [BN]). Since $K * K^\vee$ is perverse and semisimple $\mathcal{H}^0(K * K^\vee)_x \neq 0$ implies that δ_x is a perverse summand of $K * K^\vee \cong K * K = d \cdot K$. Hence $x \in H$ and this proves our claim. Together with lemma 13 we therefore obtain

Lemma 11. *For a semisimple d -regular perverse sheaf K on a complex abelian variety X , with G and $H = G^{ab}$ as above, the group H can be identified with the finite subgroup of torsion points x in $X(\mathbb{C})$ for which $T_x^*(K) \cong K$ holds; furthermore there exists a perverse sheaf K' on $X' = X/H$ such that K is the pullback of K' .*

The last lemma $K = f^*(K')$ implies $Rf_*(K) = \bigoplus_{\chi \in H^*} K' \otimes L_\chi$ for the isogeny $f : X \rightarrow X' = X/H$, and hence $f^*(Rf_*(K)) \cong \#H \cdot K$. Observe that $Rf_*(K)$ is d -regular, since K is d -regular and Rf_* is a tensor functor. For $x \in X'$ we claim

$$T_{x'}^*(Rf_*(K)) \cong Rf_*(K) \implies x' = 0 .$$

Proof of the claim. Choose $x \in X$ such that $f(x) = x'$. Then $f \circ T_x = T_{x'} \circ f$ implies $f^*(T_{x'}^*(K')) = T_x^*(f^*(K')) = T_x^*(K)$ and hence $\bigoplus_{\chi} T_{x'}^*(K') \otimes L_\chi = Rf_*(T_x^*(K))$, by applying the direct image functor. On the other hand, $Rf_*(T_x^*(K)) = T_{x'}^*(Rf_*(K))$ is by assumption isomorphic to $Rf_*(K) = Rf_*(f^*(K')) = \bigoplus_{\chi} K' \otimes L_\chi$. Since the pullbacks $f^*(L_\chi)$ for the relevant χ are all trivial, from $\bigoplus_{\chi} T_{x'}^*(K') \otimes L_\chi \cong \bigoplus_{\chi} K' \otimes L_\chi$ we get the assertion $\#H \cdot T_x^*(K) \cong \#H \cdot K$ or $T_x^*(K) \cong K$ by pullback. This implies $x \in H$ and therefore $f(x) = x' = 0$. \square

The tensor category \mathcal{T}' generated by the d -regular semisimple perverse sheaf $Rf_*(K)$ defines a finite Tannaka group denoted G' . By lemma 11 and the proof of the last claim G' is a perfect group

$$G' = [G', G'] .$$

By lemma 1 there exists an injective group homomorphism $G' \hookrightarrow G$ so that G' is a subgroup of $[G, G]$.

Lemma 12. *The group G' is perfect, and $G = G^{ab}$ holds if and only if G' is trivial.*

Proof. If $G = G^{ab}$ is abelian, then G' is trivial by perfectness. Conversely suppose G' is trivial. Then $Rf_*(K) \cong r \cdot \mathbf{1}$ and $Rf_*(K) = Rf_*(f^*(K')) = \bigoplus_{\chi} K' \otimes L_{\chi}$ implies $K' = s \cdot \mathbf{1}$ and therefore $K = s \cdot \bigoplus_{x \in H} \delta_x$. Thus $G \cong H$ is abelian. \square

Corollary 3. *If $\pi_0(G(X))$ is nonabelian, then $\pi_0(G(X'))$ admits a nontrivial simple finite nonabelian quotient group H .*

8 From \mathbb{C} to finite fields

For a complex abelian variety X and a simple (nonabelian) group H of order d with irreducible representations of order d_i for $i = 1, \dots, h$ every semisimple perverse sheaf $K = \bigoplus_{i=1}^h d_i \cdot K_i \in \text{Perv}(X, \mathbb{C})$ with $K * K \cong dK \cong dK^{\vee}$ and irreducible perverse K_i , whose associated convolution Tannakian category is isomorphic to $\text{Rep}_{\mathbb{C}}(H)$, will be called a strongly H -regular perverse sheaf on X .

We want to show the following assertion: For a complex abelian variety X and a finite simple group H of order d any strongly H -regular perverse sheaf on X is a skyscraper sheaf concentrated in torsion points of $X(\mathbb{C})$. For abelian groups H this was shown in [KrW]. So by lemma 11 it remains to consider the case of simple nonabelian groups H . We claim: *There do not exist strongly H -regular clean perverse sheaves for nonabelian simple groups H .*

For the proof of this claim we use Drinfeld's approach [Dr] to produce from a counterexample over \mathbb{C} an analogous counterexample for an abelian variety defined over a finite field κ . This is done as follows. Suppose there exists a counterexample K over \mathbb{C} , i.e. a strongly H -regular clean perverse sheaf $K \in \text{Perv}(X, \mathbb{C})$

for a simple nonabelian group H . Then the irreducible perverse constituents K_i become smooth on some open dense smooth subsets U_i of the support Y_i of K_i . There exists a scheme $I = \prod_{i=1}^h \text{Irr}_{d_i}^{U_i}$ of finite type over $\text{Spec}(\mathbb{Z})$ representing the sheaf functor on the category of commutative rings A associated to the presheaf functor $A \mapsto \underline{I}(A) = \prod_{i=1}^h \underline{\text{Irr}}_{d_i}^{U_i}(A)$, where $\underline{\text{Irr}}_d^U(A)$ is the set of isomorphism classes of rank d locally free sheaves of A -modules N on $U(\mathbb{C})$ such that $N \otimes_A k$ is irreducible for every field k equipped with a homomorphism $A \rightarrow k$. If A is a local complete ring with finite residue field, for example the completion of a local ring of a closed point of I , then $\underline{\text{Irr}}_{d_i}^{U_i}(A) \cong \text{Irr}_{d_i}^{U_i}(A)$.

Consider the ‘bad’ subset $B^\mathbb{Q} \subset I \otimes \mathbb{Q}$ of F -points (for extension fields F of \mathbb{Q}) of strongly H -regular clean perverse sheaves $K = \bigoplus_{i=1}^h d_i \cdot K_i$, where K_i is the perverse intermediate extension of the smooth perverse sheaf on $Y_i \subset X$ defined by the underlying étale sheaf of F -modules M_i , where $\prod_i M_i \in I(F)$. Using [Dr, section 3] one shows, in a similar way as for the proof of lemma 2.5. in loc. cit. (using [Dr], lemma 3.10, lemma 3.11 and the proof of lemma 3.1) that $B^\mathbb{Q}$ is a constructible subset of $I \otimes \mathbb{Q}$. So define B as the Zariski closure of $B^\mathbb{Q}$ in I . Since by assumption there exists a counterexample, we have $B^\mathbb{Q} \neq \emptyset$ and there exists a Zariski open subset $V \subseteq I$ such that $B^\mathbb{Q} \cap (V \otimes \mathbb{Q})$ is closed in $V \otimes \mathbb{Q}$ and $B \cap V$ is nonempty and smooth over \mathbb{Z} . For any closed point $z \in B \cap V$ consider the completion \widehat{I}_z of I at z and the locus \widehat{B}_z defined by B in \widehat{I}_z ; the complete local ring A_z of \widehat{B}_z has a finite residue field κ_l . Obviously the closed point z can be chosen in such a way that κ_l is a finite field of arbitrary characteristic $l \geq l_0$. In particular we can assume $l > d = \#H$.

Choose a suitable finitely generated field E with algebraic closure $\overline{E} \subset \mathbb{C}$ over which U_i, Z_i, X , can be defined so that z is $\text{Gal}(\overline{E}/E)$ -invariant. Then $\text{Gal}(\overline{E}/E)$ acts on A_z , i.e. \widehat{I}_z . If E is chosen big enough, then $\text{Gal}(\overline{E}/E)$ acts on \widehat{B}_z (as in [Dr, lemma 2.7]). Recall the key point of the argument in [Dr]: The fixed point locus of F^k on \widehat{B}_z of any Frobenius substitution $F = F_\kappa$ for closed points v of a model of X with residue field κ defined over a finitely generated ring R with quotient field E , is finite and flat over \mathbb{Z}_l and nonempty for k large enough. To show this Drinfeld uses de Jong’s conjecture (proven in [BK], [G]) which implies that the reduction mod l of the fixed point scheme is finite over κ_l and from which then easily follows the assertion above (as in [D, lemma 2.8]). If nonempty, this fixed point scheme defines an l -adic perverse sheaf \tilde{K} on a model of X defined over a suitable localizations of R with closed point v , so that \tilde{K} and its reduction \tilde{K}_0 defined over some finite extension of the residue field $\kappa = \kappa_q$ of v (of characteristic p) are ‘bad’, i.e. satisfy $K * K \cong d \cdot K \cong d \cdot K^\vee$ without being skyscraper sheaves such

that their associated Tannaka group is H . This part of the argument follows as in [BBD, section 6] and [D, section 4 and 6]. The residue characteristic $p = \text{char}(\kappa)$ of the point v can be chosen arbitrarily large $p \geq p_0$, since v can be chosen to be an arbitrary closed point of the spectrum of the ring R , which is finitely generated over \mathbb{Z} ; in fact we can choose $p > d = \#H$ and $l \neq p$ suitably.

This construction of [D] and [BBD] reduces the proof of the characteristic zero assertion $\text{Reg}(n)$ for strongly d -regular clean perverse sheaves $K \in \text{Perv}(X, \mathbb{C})$ to the proof of the corresponding assertion for strongly d -regular clean perverse l -adic sheaves K_0 on abelian varieties over finite fields κ . In positive characteristic a semisimple perverse sheaf K is called clean, if no character twist K_χ contains acyclic irreducible constituents (over \mathbb{C} this is equivalent to the previous notion).

So to complete the proof of theorem 2 via the induction argument using proposition 1, for the relevant conclusion $\text{Reg}(n) \implies \text{Tor}(n)$ (see section 6) it will be enough to show the following

Theorem 4. *Let X_0 be an abelian variety over a finite extension field of a finite field κ of characteristic p and let K_0 be a perverse $\overline{\mathbb{Q}}_l$ -adic sheaf on X_0 for $l \neq p$. Let X be the scalar extension of X_0 to k , and let K be the extension of K_0 to a perverse sheaf on X . Then K is not a strongly H -regular clean semisimple perverse sheaf on X for some finite simple nonabelian group H .*

Proof. Let us outline the arguments for theorem 4 and the characteristic p arguments. The details will be given in the subsequent sections 9, 11 and 13.

Suppose given some l -adic perverse sheaf K_0 whose scalar extension K is strongly H -regular. We can assume that H is a simple nonabelian group. In a first reduction step (section 9) we show that we then can assume that K_0 itself is strongly H -regular. By lemma 14 we can assume that the coefficients of K_0 are contained in a finite extension field E_λ of \mathbb{Q}_l . Then by lemma 14 and by a theorem of Deligne [D] there exists a number field E (in our case this is a cyclotomic field) such that the supertraces $\text{Tr}(F_x^k, K_x)$ of the powers of the Frobenius substitution F_x for the closed points x of X are contained in E . Hence by theorem 5, which is a version of a theorem of Drinfeld [Dr2, thm.1.1], and the comments on conjecture [Dr2, conj.1.3(b)] for any prime l' different from l and the characteristic p of the field κ , there exists a prime λ' of E (possibly a finite extension field of E) over l' and a $E_{\lambda'}$ -adic perverse sheaf K'_0 on X_0 compatible with K_0 , i.e. the supertraces $\text{Tr}(F_x^k, K'_x)$ of the powers of all Frobenius substitutions F_x coincide for the closed points x of X .

Then we consider the Fourier transform of the functions $Tr(F_x^k, K_{\bar{x}})$ and show that this Fourier transform is locally constant using an l -adic deformation argument. This severely restricts the support of the functions $Tr(F_x^k, K_{\bar{x}})$, so that it is easy to show that this support must be finite.

9 H -regular descent to finite fields

Suppose X_0 is an abelian variety defined over a finite field κ , and let X be its scalar extension over the algebraic closure $k = \bar{\kappa}$. Let $K_0 \in Perv(X_0, E_\lambda)$ be a perverse sheaf with coefficients in E_λ ; we then also view it as a perverse sheaf with coefficients in $\Lambda = \overline{\mathbb{Q}}_l$. Suppose that the scalar extension $K \in Perv(X, \Lambda)$ of K_0 is a strongly H -regular semisimple clean perverse sheaf for a nonabelian simple group H .

Since $K \cong \bigoplus_{i=1}^h d_i \cdot K_i$ for simple perverse sheaves K_i , without restriction of generality we can assume that each K_i is defined over κ by suitably enlarging κ if necessary. By a choice of certain absolutely irreducible perverse sheaves $K_{0,i}$ on X_0 whose extension to X is K_i within each of the h isomorphism classes, we define

$$K_0 = \bigoplus_i d_i \cdot K_{0,i}$$

so that $K_{0,1} = \delta_0$ and

$$K_{0,i} \cong K_{0,j} \iff K_i \cong K_j .$$

Then, for the induced Weil sheaf structure on K , the Frobenius morphism $Fr = Fr_\kappa$ acts trivially on $End_{Perv(X, \Lambda)}(K)$. The perverse $K_0 \oplus K_0^\vee$ generates a convolution tensor subcategory \mathcal{T}_0 of $Perv(X_0, \Lambda)$. Since the Euler characteristic of each irreducible object K_i is nonnegative for a strongly H -regular perverse sheaf K and since $\Lambda = \overline{\mathbb{Q}}_l$ is an algebraically closed field, this is a Tannaka category. Let G be its Tannaka group.

The indecomposable elements in $Perv(X_0, \Lambda)$ are of the form $S_0(\gamma) \otimes_\Lambda E_n$, where E_n is a Λ -vectorspace of dimension n on which Fr_κ acts by a nilpotent matrix with one Jordan block ([BBD], p.139) and where S_0 is an absolute simple perverse

sheaf with determinant of finite order and $\gamma \in \Lambda^*$ defines a generalized Tate twist (see page 33 for further details). Obviously then $K_{0,i} \cong S_{0,i}(\gamma_i)$ for some $\gamma_i \in \Lambda^*$ and

$$K_0 * K_0 \cong \bigoplus_i \bigoplus_j S_{0,i}(\beta_{ij}) \otimes_{\Lambda} E_{n(i,j)}$$

$$K_0^{\vee} \cong \bigoplus_i S_{0,i}(\beta_i) \otimes_{\Lambda} E_{n(i)} .$$

for certain $\beta_{ij} \in \Lambda^*$ and $\beta_i \in \Lambda^*$. Replacing κ by a suitable finite field extension, we get rid of all Jordan blocks. The Tate twists $\delta_0(\beta_{ij}), \delta_0(\beta_i)$ of the unit perverse sheaf δ_0 generate a tensor category \mathcal{T}_{Weil} , since $L^* \delta_0(\beta) = L(\beta)$ and $\delta_0(\beta) * \delta_0(\beta') = \delta_0(\beta\beta')$. All irreducible objects in this tensor category \mathcal{T}_{Weil} are therefore invertible. Hence the Tannaka group G_{Weil} of \mathcal{T}_{Weil} is a diagonalizable commutative algebraic group over Λ , and hence there exists an exact sequence

$$0 \rightarrow (\mathbb{G}_m)^{\vee} \rightarrow G_{Weil} \rightarrow \pi_0(G_{Weil}) \rightarrow 0 .$$

This sequence is a split exact sequence by the structure theory of diagonalizable groups. Thus $G_{Weil} \cong (\mathbb{G}_m)^{\vee} \times \pi_0(G_{Weil})$. Since \mathcal{T}_{Weil} is a tensor subcategory of \mathcal{T}_0 there exists a surjective homomorphism

$$p : G \rightarrow G_{Weil} .$$

Any irreducible object in \mathcal{T}_0 is of the form $S_{0,i} * \delta_0(\alpha)$ for some $\alpha \in \Lambda^*$. Extension of scalars from κ to its algebraic closure k defines an exact faithful tensor functor from \mathcal{T}_0 to $Rep_{\Lambda}(H)$. Since by construction any representation of H is in the image, this functor by Tannaka duality [DM, prop 2.21], [D2] induces an injective group homomorphism

$$res : H \hookrightarrow G .$$

A representation of G becomes trivial on H if and only if it is contained in the subcategory \mathcal{T}_{Weil} . This shows $p \circ res = 0$ and $H \subseteq H' = Kern(p)$. But now obviously the restriction $Rep_{\Lambda}(H') \rightarrow Rep_{\Lambda}(H)$ induces an equivalence of tensor categories, since both categories are semisimple and have the same irreducible objects (the h nonisomorphic constituents of K_0 up to Tate twist). This implies $H \cong H'$ and therefore $H = Kern(res)$. Hence G is a reductive group over Λ with connected component $G^0 \cong (\mathbb{G}_m)^{\vee}$. Furthermore its group $\pi_0(G) = G/G^0$ of Zariski connected components sits in an exact sequence

$$0 \rightarrow H \rightarrow \pi_0(G) \rightarrow \pi_0(G_{Weil}) \rightarrow 0 .$$

The tensor category of \mathcal{T}_0 generated by the representations of $\pi_0(G_{Weil})$ is generated by the invertible objects $\delta_0(\beta)$ where $\beta \in \Lambda^*$ is a root of unity. In fact there must exist an integer n such that $\beta^n = 1$, since otherwise $\delta_0(\beta)^{*n} \cong \delta_0(\beta^n) \not\cong \delta_0$ would give infinitely many irreducible non isomorphic representations of the finite group $\pi_0(G_{Weil})$. If we pass from κ to a finite extension of degree n , then $\delta_0(\beta)$ is replaced by $\delta_0(\beta^n)$. Hence after a suitable finite field extension $\pi_0(G_{Weil})$ becomes trivial and without restriction of generality we can assume $\pi_0(G) = H$. Hence the connected component G^0 of G is isomorphic to $(\mathbb{G}_m)^\vee$ and we have an exact sequence

$$0 \rightarrow (\mathbb{G}_m)^\vee \rightarrow G \rightarrow H \rightarrow 0$$

which splits by the inclusion $res : H \rightarrow G$ if H is a simple noncyclic finite group

$$G \cong (\mathbb{G}_m)^\vee \triangleleft H.$$

Indeed $res(H) \cap (\mathbb{G}_m)^\vee$ defines a normal abelian subgroup of the simple group H and hence is trivial. Hence over a suitable finite extension field of κ the tensor category \mathcal{T}_0 contains a strongly H -regular tensor subcategory \mathcal{T}'_0 . Its regular representation K'_0 becomes isomorphic to K over the algebraic closure over κ . But then

$$K'_0 \cong \bigoplus_{i=1}^h d_i \cdot S_{0,i}(\alpha_i)$$

holds for certain $\alpha_i \in \Lambda^*$, since otherwise \mathcal{T}'_0 would contain perverse sheaves of the form $(S_{0,i})^\vee * S_{0,i}(\alpha)$, and hence there would exist $\delta_0(\alpha) \in \mathcal{T}'_0$ for some $\alpha \neq 1$ which is excluded. Hence for a suitable extension κ of the original base field, we may assume $K_0 = K'_0$ over κ . This proves part of the next

Lemma 13. *Let H be a finite simple group. Then for a strongly H -regular perverse sheaf K in $Perv(X, \overline{\mathbb{Q}}_l)$ on an abelian variety X over k that is defined over a finite subfield of k , there exists a strongly H -regular pure perverse sheaf K_0 of weight zero in $Perv(X_0, \overline{\mathbb{Q}}_l)$ defined over a finite subfield κ whose pullback to k is K , such that F_κ acts trivial on $End_{Perv(X, \Lambda)}(K)$.*

Proof. Looking at the weight filtration the property $K_0 * K_0 \cong d \cdot K_0$ immediately implies that K_0 is pure of weight zero. \square

We say $\alpha \in \Lambda^*_{mot}$, if α is contained in a finite number field $E \subset \Lambda$ and α, α^{-1} are integral over $\mathbb{Z}[p^{-1}]$, where p is the characteristic of κ . In the following we say that $\alpha, \alpha' \in \Lambda^*$ are equivalent, if α'/α is a root of unity. For an irreducible perverse Weil

sheaf P on X with respect to κ then $P(\alpha')$ and $P(\alpha)$ become isomorphic as perverse Weil sheaves over some finite extension field of κ iff α' and α are equivalent. If P is the scalar extension of a perverse sheaf P_0 over κ , then there exists a constant $\alpha \in \Lambda^*$ such that for $P_0 = S_0(\alpha)$ the determinant of the underlying smooth coefficient system of S_0 has finite order; and up to equivalence α is uniquely defined by P_0 . Then by [L, cor. VII.8] the perverse sheaf S_0 is pure of weight zero.

In the proof of the last lemma the choice of the perverse sheaves K_{0i} made at the beginning was not unique. However by lemma 13 after a finite extension of κ there exists a choice so that $K_0 = \bigoplus_i d_i \cdot K_{0i}$ is strongly H -regular. Hence $K_{0i} * K_{0j} \cong \bigoplus_k c_{ij}^k \cdot K_{0k}$ for the Clebsch-Gordan coefficients of H and $K_{0i}^\vee \cong K_{0\tilde{i}}$ for a unique \tilde{i} depending only on i . If we replace K_{0i} by $K_{0i}(\alpha_i^{-1})$ for some $\alpha_i \in \Lambda^*$, then in general $K_0 = \bigoplus_i d_i \cdot K_{0,i}(\alpha_i^{-1})$ generates a Tannakian category with Tannaka group $G \neq H$. A particular interesting case is the twist $K_{0i}(\alpha_i^{-1}) = S_{0i}$. For a simple group H every nontrivial irreducible representation K_{0i} of H generates the Tannakian representation category and a high enough tensor power K_{0i} contains the trivial representation δ_0 of H , e.g. $(K_{0i})^{*n}$ for $n = \dim(K_{0i})$ contains the one dimensional and hence trivial representation $\Lambda^n(K_{0i})$. Hence $\delta_0 \hookrightarrow K_{0i}^{*n}$ holds for some integer n , and then $\delta_0(\alpha_i^{-n}) \hookrightarrow S_{0i}^{*n}$ implies $\alpha_i^n \in \Lambda_{mot}$ using [Dr2, thm. B3]. Hence also $\alpha_i \in \Lambda_{mot}$.

Lemma 14. *With the notations of the last lemma $K_{0i} \cong S_{0i}(\alpha_i)$ holds for $i = 1, \dots, h$ with twist coefficients $\alpha_i \in \Lambda_{mot}$ of weight zero.*

10 Cebotarev density theorem

Suppose X_0 is a variety defined over a finite field κ . Let k denote the algebraic closure of κ and let X be the extension of X_0 to k . On X_0 the geometric Frobenius endomorphism $Fr = Fr_\kappa$ acts by the q -power map on coordinates, for $q = \#\kappa$.

Fix $\Lambda = \overline{\mathbb{Q}_l}$ and some isomorphism $\tau : \Lambda \cong \mathbb{C}$, which allows to define complex conjugation on Λ . Usually we suppress to write τ and write $\overline{\alpha}$ instead of $\tau^{-1}\tau(\alpha)$. Let K be a perverse Λ -adic Weil sheaf on X , which is τ -pure of integral weight w . This means that K is equivariant with respect to the Frobenius F_X so that exists an isomorphism $F^* : F_X^*(K) \cong K$. If K is an irreducible perverse sheaf on X , then two such Weil sheaf structures (K, F_1^*) and (K, F_2^*) on K yield an automorphism

$F_2^* \circ (F_1^*)^{-1} : K \cong K$ which is given by $\alpha \cdot id_K$ for some $\alpha \in \Lambda^*$, since $End_{Perv(X, \Lambda)}(K) = \Lambda \cdot id_K$.

Although we are only interested in the case, where K is an irreducible perverse sheaf obtained from a Λ -adic perverse sheaf K_0 on X_0 by extension of scalars, it is convenient to view them as Weil sheaves over X . The scalar extensions K of Λ -adic perverse sheaf K_0 on X_0 are mixed Weil sheaf by a deep result of Lafforgue [L]. If the perverse sheaf K is irreducible on X , then it is a pure Weil sheaf. By a formal half integral Tate twist we can and will always assume that the weight is $w = 0$. Two Weil sheaf structures (both of weight 0) for an irreducible perverse sheaf K on X differ by a generalized Tate twist defined by some $\alpha \in \Lambda$ such that $|\tau(\alpha)| = 1$. For simplicity we also write $|\alpha| = 1$, if τ is fixed.

Let κ_m be an extension field of κ of degree m . Attached to a Weil complex K there is the function

$$f_m^K : X(\kappa_m) \longrightarrow \Lambda$$

defined by the supertraces

$$f_m^K(x) = sTr(Fr_x^{\frac{m}{d(x)}}; K_{\bar{x}}) = sTr(Fr_{\kappa}^m; K_{\bar{x}}) ,$$

where $Fr_x \in Gal(\bar{\kappa}/\kappa_m(x))$ is the geometric Frobenius at the closed point x acting on the stalk $K_{\bar{x}}$ of K of a geometric point \bar{x} over x . If we replace κ by κ_m , we may assume $m = 1$ and then simply write $f_1^K(x) = f^K(x)$.

Let K, K' be pure perverse Weil sheaves of weight w on a variety X over k , where k is the algebraic closure of a finite field κ with q elements. The Frobenius $Fr = Fr_{\kappa}$ acts on the cohomology groups $H^{\bullet}(X, D(K) \otimes^L K')$. There exist complexes $RHom(K', K)$ of abelian groups and $R\mathcal{H}om(K', K)$ of sheaves such that $R\Gamma(X, R\mathcal{H}om(K', K)) = RHom(K', K)$ and $Hom_{D_c^b(X, \Lambda)}(K', K) = H^0(RHom(K', K))$ holds. Since by definition $Hom_{Perv(X)}(K', K) = Hom_{D_c^b(X, \Lambda)}(K', K)$, we get

$$Hom_{Perv(X)}(K', K) = H^0(R\Gamma(X, R\mathcal{H}om(K', K))) = H^0(X, R\mathcal{H}om(K', K)) .$$

Recall $R\mathcal{H}om(K', K) \in D^{\geq 0}(X, \Lambda)$ from [KW, lemma 4.3], since K and K' are perverse sheaves. On the other hand $R\mathcal{H}om(K', K) \cong D(K' \otimes^L D(K))$, so by Poincare duality

$$Hom_{Perv(X)}(K', K) = H_c^0(X, K' \otimes^L D(K))^* .$$

The support conditions for the perverse sheaves K' and $D(K)$, to be in ${}^pD^{\leq 0}(X)$, imply that the cohomology groups $H_c^i(X, K' \otimes^L D(K))$ vanish for $i > 0$. Since by

assumption K and K' are pure of weight w , we furthermore get $w(K' \otimes^L D(K)) \leq 0$. Hence

$$H_c^\bullet(X, K' \otimes^L D(K))$$

has weights ≤ 0 , and the weight 0 only occurs as eigenvalue of the zero-th cohomology group $H_c^0(X, K' \otimes^L D(K))^*$.

By twisting K and K' we may now suppose $w = 0$. By the dictionary [KW, theorem III.12.1 (6)] and the Grothendieck-Lefschetz trace formula, the characteristic function $\tau \text{trace}(Fr^m, H_c^\bullet(X, K' \otimes^L D(K)))$ is

$$(f_m^{K'}, f_m^K) := \sum_{x \in X(\kappa_m)} f_m^{K'}(x) \overline{f_m^K}(x) = \sum_{x \in X(\kappa_m)} f_m^{K' \otimes^L D(K)}(x) .$$

Hence by the vanishing result from above

$$(f_m^{K'}, f_m^K) = \text{Tr}(Fr^m; \text{Hom}_{\text{Perv}(X)}(K', K)) + \sum_{v < 0} (-1)^v \tau \text{Tr}(Fr^m; H_c^v(X, K' \otimes^L D(K))) .$$

The eigenvalues of Fr^m on $H_c^v(X, K' \otimes^L D(K))$ are $\leq q^{vm/2}$, hence the sum on the right side can be estimated by $C \cdot q^{-m/2}$ for a constant $C = \sum_{v < 0} \dim_\Lambda H_c^v(X, K' \otimes^L D(K))$ depending only on K and K' but not on m . In other words

$$|(f_m^{K'}, f_m^K) - \text{Tr}(Fr^m; \text{Hom}_{\text{Perv}(X)}(K', K))| \leq C \cdot q^{-m/2} .$$

Corollary 4. *If K and K' are pure perverse Weil sheaves of weight w , then for any $\varepsilon > 0$ there exists an integer $m_0 = m(\varepsilon, K, K')$ such that*

$$\dim_\Lambda(\text{Hom}_{\text{Perv}(X)}(K, K')) > 0$$

holds provided $(f_m^{K'}, f_m^K) > \varepsilon$ holds for some $m \geq m_0$.

Proof. Indeed $|\text{Tr}(Fr^m; \text{Hom}_{\text{Perv}(X)}(K, K')) - (f_m^{K'}, f_m^K)| < Cq^{-m/2}$. □

Corollary 5. *For an irreducible perverse sheaf of weight zero defined over a finite field κ*

$$\lim_{m \rightarrow \infty} \|K\|_m^2 = \lim_{m \rightarrow \infty} \sum_{x \in X(\kappa_m)} |f_m^K(x)|^2 = 1 .$$

11 Fourier transform

Suppose X_0 is an abelian variety defined over a finite field κ . Let $a_0 : X_0 \times X_0 \rightarrow X_0$ be the group law. Let k denote the algebraic closure of κ and let X be the extension of X_0 to k . On X_0 the geometric Frobenius *endomorphism* $Fr = Fr_\kappa$ acts by the q -power map on coordinates for $q = \#\kappa$.

Lang torsors. The Lang torsor for X_0 is defined by the etale homomorphism $\wp_0(x) = Fr(x) - x$

$$\wp_0 : X_0 \rightarrow X_0 .$$

The morphism \wp_0 defines a finite etale geometrically irreducible Galois covering of X_0 whose Galois group by [S2, p.116] is the abelian finite group $X_0(\kappa)$, i.e. the kernel of \wp_0 . Considered over the algebraic closure k of κ , this defines an etale covering with Galois group Δ and will be denoted $\wp : X \rightarrow X$

$$0 \longrightarrow \Delta \longrightarrow X \xrightarrow{\wp} X \longrightarrow 0 .$$

Since \wp_0 is geometrically irreducible, we get $\Delta = X_0(\kappa)$. Let Δ^* denote the group of characters of Δ with values in Λ^*

$$\chi : \Delta \longrightarrow \Lambda^* .$$

The direct image $\wp_{0,*}(\Lambda_{X_0})$ of the constant sheaf Λ_{X_0} on X_0 decomposes into a direct sum $\bigoplus_{\chi \in \Delta^*} L_{\chi,0}$ of smooth rank one Λ -adic sheaves $L_{\chi,0}$ on X_0 . Let denote L_χ the scalar extension of $L_{\chi,0}$ to k , defining the corresponding smooth etale Weil sheaf on X . By class field theory, see [S2, p.142],

$$\boxed{Tr(Fr_x^m, L_\chi) = \chi(x)^{-m}}$$

holds for all points $x \in X(\kappa)$. Hence the functions

$$f^{L_\chi}(x) = \chi(x)^{-1}$$

separate points in $X(\kappa)$.

Let $m \geq 1$ be an integer. Then $Fr^m(x) - x$ again defines a geometrically irreducible etale morphism $X_0 \rightarrow X_0$, which however becomes a Galois covering only after a base field extension by passing to the finite extension field κ_m of

κ of degree m , where it induces the Lang torsor $\wp_0^{(m)} : X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m) \rightarrow X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m)$ of the abelian variety $X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m)$ over $\text{Spec}(\kappa_m)$

$$\begin{array}{ccc}
 & X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m) & \\
 \swarrow & \downarrow \wp_0^{(m)} & \\
 X_0 & & X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m) \\
 \downarrow Fr^m - id & \swarrow & \downarrow \\
 X_0 & & \text{Spec}(\kappa_m) \\
 \downarrow & \swarrow & \\
 \text{Spec}(\kappa) & &
 \end{array}$$

The Frobenius *automorphism* $F = F_\kappa$ acts on $X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m)$ via its Galois action on κ_m inducing an action of F on $X_0(\kappa_m) = \text{Hom}_{\text{Spec}(\kappa_m)}(\text{Spec}(\kappa_m), X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m))$. This action of F coincides with the action of the Frobenius endomorphism Fr on $X_0(\kappa_m)$. By $F^m - 1 = (F - 1) \sum_{i=0}^{m-1} F^i$ the trace $\sum_{i=0}^{m-1} F^i$ defines a homomorphism

$$S_m = \sum_{i=0}^{m-1} F^i : X_0(\kappa_m) \longrightarrow X_0(\kappa)$$

which is surjective by [S2, VI, §1.6]. Any character $\chi : X(\kappa) \rightarrow \Lambda^*$ can be extended to a character

$$\chi_m : X(\kappa_m) \longrightarrow \Lambda^*$$

where $\chi_m = \chi \circ S_m$ is defined by the composite of the trace $S_m : X(\kappa_m) \rightarrow X(\kappa)$ and the character χ . Then by definition $\chi_m(x) = \chi(x^m) = \chi(x)^m$ holds for $x \in X(\kappa)$.

More generally $S_{r,m} : X(\kappa_r)^* \rightarrow X(\kappa_{rm})^*$ defined by $\chi_r \mapsto \chi_r \circ S_{r,m} = \sum_{i=0}^{m-1} Fr^i$ is injective. Using these transition maps any character $\chi_r \in X(\kappa_r)^*$ defines a collection of characters $(\chi_{rm})_{m \geq 1}$ such that $\chi_{rm} \in X(\kappa_{rm})^*$. Any such $(\chi_{rm})_{m \geq 1}$ defines a translation invariant sheaf, say L_ψ , on X . In this sense we view ψ as a character on all $X_0(\kappa_{rm})$ for $m = 1, 2, \dots$. For large enough n all L_ψ for torsion characters χ of $\pi_1(X, 0) \rightarrow \Lambda^*$ arise in this way. Indeed for $\chi^k = 1$ chose r large enough so that $X[k] \subset X(\kappa_r)$. Then there exist etale isogenies $f : X' \rightarrow X$ and $g : X \rightarrow X'$ such that $f_s \circ g = \wp^{(r)}$, where $f_s : X' \rightarrow X$ is the etale part of the isogeny $k \cdot id_X = f_s \circ f_{ins}$ and f_{ins} is the inseparable part. Then as a constituent of $f_{s*}(\Lambda_{X'})$ the sheaf L_ψ is also a direct summand of $\wp_*^{(r)}(\Lambda_X)$, since $\Lambda_{X'}$ is a direct summand of $g_*(\Lambda_X)$.

The Frobenius automorphism F acts on each $X_0(\kappa_n)$ and hence on the characters $\psi : X_0(\kappa_n) \rightarrow \Lambda^*$, so that F^n acts trivially. Conversely suppose $\psi^{F^r} = \psi$ or $\psi((F^r - id)(x)) = 1$ holds for all $x \in X_0(\kappa_n)$ and all n . We may enlarge n and hence assume that r divides n , hence $n = rm$. We may replace κ by κ_r and F^r by F_{κ_r} . Then ψ factorizes over the quotient $X_0(\kappa_n)/(1 - F_{\kappa_r})X_0(\kappa_n)$ which is isomorphic to $X_0(\kappa_r)$ via the trace homomorphism. Hence ψ comes from a character $\psi' : X_0(\kappa_r) \rightarrow \Lambda^*$ by the trace extension $\psi = \psi'_{n/r}$ defined above.

Induction. For a finite field extension κ_r of κ and a character $\psi : X_0(\kappa_r) \rightarrow \Lambda^*$ we consider the Weil sheaf L_ψ on X , i.e. it is F_{κ_r} -equivariant so that there exists an isomorphism $F_{\kappa_r}^*(L_\psi) \cong L_\psi$ for the Frobenius automorphism F_{κ_r} of the field κ_r . Let L be an F_{κ_r} -equivariant Weil sheaf on X . Since $F_{\kappa_r} = F^r$ holds for the Frobenius automorphism F for the field κ , $K = \bigoplus_{i=0}^{r-1} (F^i)^*(L_\psi)$ defines an F -equivariant Weil sheaf on X over κ , i.e. one has an isomorphism $F^*(K) \cong K$ and we write $K = Ind_{\kappa_r}^\kappa(L)$. Any F -equivariant Weil sheaf K on X , which as a perverse sheaf on X is translation invariant under X and multiplicity free, is of the form $K \cong \bigoplus_\psi L_\psi(\alpha_\psi)$ for certain characters $\psi : X_0(\kappa_r(\psi)) \rightarrow \Lambda^*$ and certain Tate twists by $\alpha_\psi \in \Lambda^*$, where we can assume that the integers $r(\psi)$ are chosen minimal. Then $F^*(K) \cong K$ implies for certain $\varphi \in \{\psi\}$

$$K \cong \bigoplus_{\varphi} Ind_{\kappa_r}^\kappa(L_\varphi(\alpha_\varphi))$$

so that $F^i(L_\varphi) \cong L_\varphi$ iff $r(\varphi)$ divides i .

Character Twists. For perverse sheaves K_0 on X_0 and $\chi_0 \in \Delta^*$ we can define the twisted complex $K_0 \otimes L_{0,\chi}$, which again is a perverse sheaf on X_0 . Let $K \otimes \chi$ or K_χ denote the corresponding perverse Weil complex on X . Its associated function (over κ_m) is

$$f_m^{K \otimes \chi}(x) = f_m^K(x) \cdot \chi_m(x)^{-1}.$$

Fourier transform. By varying the characters $\chi : X(\kappa_m) \rightarrow \Lambda^*$ we define the Fourier transform \widehat{f}_m^K of f_m^K for each m by the following summation

$$\widehat{f}_m^K : X(\kappa)^* \rightarrow \Lambda.$$

$$\boxed{\widehat{f}_m^K(\chi) = \sum_{x \in X(\kappa_m)} f_m^K(x) \cdot \chi_m(-x)}.$$

Obviously this Fourier transform is additive with respect to the perverse sheaf K , in the sense that

$$\widehat{f}_m^{K \oplus L}(\chi) = \widehat{f}_m^K(\chi) + \widehat{f}_m^L(\chi).$$

Example 1. For the skyscraper sheaf $K_0 = \delta_0$ concentrated at the origin the Fourier transform is constant, i.e. $\widehat{f}_m^K(\chi) = 1$ holds for all m and all χ .

Example 2. The Fourier transform $\widehat{f}_m^K(\chi)$ for the character sheaf $K_0 = L_\varphi$ of a character $\varphi : X_0(\kappa_m) \rightarrow \Lambda$ is $\#X(\kappa_m)$ for $\chi = \varphi^{-1}$, and it is zero otherwise. The perverse sheaf $K = \delta_X^\varphi$ for

$$\delta_X^\varphi := L_\varphi[\dim(X)]\left(\frac{\dim(X)}{2}\right)$$

is of weight $w = 0$. Its Fourier transform $\widehat{f}_m^K(\chi)$ is $(-1)^{\dim(X)} q^{-m \dim(X)/2} \#X(\kappa_m)$ for $\chi = \varphi^{-1}$, and zero otherwise.

Example 3. Given $\psi : X_0(\kappa_r) \rightarrow \Lambda^*$ for an extension κ_r of κ of degree r suppose for the Frobenius automorphism F of κ that $F^i(L_\psi) \cong L_\psi$ holds iff r divides i . Then $K = \text{Ind}_{\kappa_r}^\kappa(L_\psi)$ is defined over κ . If r divides m , then $f_m^K(x) = \sum_{i=0}^{r-1} f_m^{F^i(L_\psi)}(x)$ and hence f_m^K can be computed by example 2); otherwise

$$f_m^K(x) = 0 \quad \text{for} \quad r \nmid m.$$

By base extension this last assertion can be easily reduced to the case $m = 1$. For $m = 1$ then $f^K(x) = \text{Tr}(Fr_x; K_{\bar{x}})$ holds for $x \in X_0(\kappa)$. Since the substitution Fr_x permutes the summands of $K_{\bar{x}} = \bigoplus_{i=0}^{r-1} F^i(L_\psi)_{\bar{x}}$, in the sense that $Fr_x : F^i(L_\psi)_{\bar{x}} \rightarrow F^{i+1}(L_\psi)_{\bar{x}}$ for $i < r-1$ and $Fr_x : F^{r-1}(L_\psi)_{\bar{x}} \rightarrow F^r(L_\psi)_{\bar{x}} \cong (L_\psi)_{\bar{x}}$, the trace of Fr_x is zero unless $r = 1$.

By the Grothendieck-Lefschetz formula the Fourier transform also has the following interpretation

$$\widehat{f}_m^K(\chi) = \sum_{x \in X(\kappa_m)} f_m^{K \otimes \chi}(x) = f_m^{Rp_*(K \otimes \chi)}(*)$$

for the structure map $p_0 : X_0 \rightarrow \text{Spec}(\kappa) = \{*\}$. In other words $f_m^{Rp_*(K \otimes \chi)}(*)$ is the trace of Fr^m on the etale cohomology group $H^\bullet(X, K_\chi)$.

Convolution. For Λ -adic Weil complexes K and L on X the convolution $K * L$ is the direct image complex $K * L = Ra_*(K \boxtimes L)$ for the group law $a : X \times X \rightarrow X$. Hence by the Grothendieck etale Lefschetz trace formula

$$f_m^{K * L}(x) = \sum_{y \in X(\kappa_m)} f_m^K(x - y) f_m^L(y)$$

which is the usual convolution of the functions $f_m^K(x)$ and $f_m^L(x)$ on the finite abelian group $X(\kappa_m)$. By elementary Fourier theory

$$\widehat{f_m^{K*L}}(\chi) = \widehat{f_m^K}(\chi) \cdot \widehat{f_m^L}(\chi)$$

$$\widehat{f_m^{K^\vee}}(\chi) = \overline{\widehat{f_m^K}(\chi)},$$

where the second equality follows from [KW], theorem III.12.1(6) and the assumption that K is pure of weight $w = 0$. Indeed

$$\widehat{f_m^{K^\vee}}(\chi) = \sum_x f_m^{D(K)}(-x) \chi_m(-x) = \sum_x \overline{f_m^K(-x)} \chi_m(-x) = \sum_x \overline{f_m^K(x)} \chi_m(x) = \overline{\widehat{f_m^K}(\chi)}.$$

For $y \in X(\kappa_m)$ one has the *Fourier inversion formula*

$$f_m^K(y) = \frac{1}{\#X(\kappa_m)} \sum_{\chi \in X(\kappa_m)^*} \widehat{f_m^K}(\chi) \chi_m(y)$$

with summation over all characters $\chi : X(\kappa_m) \rightarrow \Lambda^*$ (i.e. the characters obtained from the Lang torsor). If K is pure of weight zero, in the limit $m \rightarrow \infty$ the sum

$$\|K\|_{X_m}^2 = \sum_{x \in X(\kappa_m)} |f_m^K(x)|^2 = (f_m^K, f_m^K)$$

converges to $Tr(Fr^m; End_{Perv(X)}(K))$ as shown in section 10.

12 The Plancherel formula

Let X be an abelian variety X over the algebraic closure k of a finite field κ . Let K be a pure perverse Weil sheaf of weight 0 on X , so K is equivariant $F^*(K) \cong K$ for the Frobenius automorphism $F = F_\kappa$. The elementary Plancherel formula expresses $\|K\|_{X_m}^2 = \|f_m^K\|^2$ in terms of the Fourier transform $\widehat{f_m^K}(\chi)$ of $f^K(x)$

$$\|K\|_{X_m}^2 = \frac{1}{\#X(\kappa_m)} \cdot \|\widehat{f_m^K}\|^2.$$

Here by definition the L^2 -norms are $\|f_m^K\|^2 = (f_m^K, f_m^K)$ for $(f, g) = \sum_{x \in X(\kappa_m)} f(x) \overline{g(x)}$ resp. $\|\widehat{f_m^K}\|^2 = (\widehat{f_m^K}, \widehat{f_m^K})$ for $(\widehat{f}, \widehat{g}) = \sum_{\chi \in X(\kappa_m)^*} \widehat{f}(\chi) \overline{\widehat{g}(\chi)}$. More generally

$$(f_m^K, f_m^L) = \frac{1}{\#X(\kappa_m)} \cdot (\widehat{f_m^K}, \widehat{f_m^L}).$$

Example. (f_m^K, f_m^L) for the Weil sheaves $K = L_\psi(\alpha)$ and $L = L_\phi(\beta)$ vanishes unless $\psi = \phi$, and in this case $(f_m^K, f_m^L) = \alpha \cdot \bar{\beta} \cdot \#X(\kappa_m)$. For the following recall that the perverse Weil sheaf $\delta_X^\phi = L_\phi[\dim(X)](q^{-\dim(X)/2})$ is pure of weight zero.

Lemma 15. *An irreducible perverse Weil sheaf K on X of weight zero, for which for almost all characters χ the cohomology groups $H^\bullet(X, K_\chi)$ are zero, is isomorphic to δ_X^ϕ for some character ϕ . A pure perverse Weil sheaf K of weight zero on X is translation invariant on X , if almost all cohomology groups $H^\bullet(X, K_\chi)$ are zero.*

Proof. If K is irreducible and $K \not\cong \delta_X^\phi$ holds for all characters ϕ , then $H^i(X, K_\chi) = 0$ for all χ and all i with $|i| \geq \dim(X)$. Suppose there exist only finitely many characters χ_1, \dots, χ_r for which $H^\bullet(X, K_\chi)$ does not vanish. Then $\hat{f}_m^K(\chi) = 0$ holds except for $\chi = \chi_i$ with $i = 1, \dots, r$. Furthermore there exist constants c_i independent from m so that $|\hat{f}_m^K(\chi_i)| \leq c_i \cdot q^{m(\dim(X)-1)/2}$ by the Weil conjectures. Hence $\|K\|_m \leq \frac{\text{const}}{\#X(\kappa_m)} \cdot q^{m(\dim(X)-1)/2}$ and this implies $\|K\|_m \rightarrow 0$ in the limit $m \rightarrow \infty$ contradicting corollary 5. For pure K we can apply this argument to the simple constituents P of K in order to show that K is translation invariant under X . Notice that $H^\bullet(X, K_\chi) = 0$ implies $H^\bullet(X, P_\chi) = 0$ by the decomposition theorem. \square

Corollary 6. *For an irreducible perverse Weil sheaf K on X , acyclic in the sense that $H^\bullet(X, K) = 0$ holds, there exists a torsion character $\chi : \pi_1(X, 0) \rightarrow \Lambda$ for which $H^\bullet(X, K_\chi) \neq 0$ holds.*

Lemma 16. *Suppose $X_0 = A_0 \times B_0$ is a product of two abelian varieties defined over a finite field κ . Let K be a pure Weil sheaf of weight 0 on X and let $p_0 : X_0 \rightarrow B_0$ be the projection onto the second factor. Then we have the following relative Plancherel formula*

$$\|K\|_{X_m}^2 = \frac{1}{\#A(\kappa_m)^*} \sum_{\chi \in A(\kappa_m)^*} \|Rp_*(K_\chi)\|_{B_m}^2$$

where the summation is over all characters $\chi : A(\kappa_m) \rightarrow \Lambda^*$.

Proof. Notice $\|f_m^K\|_{X(\kappa_m)}^2 = \sum_{(x,y) \in X(\kappa_m)} |f_m^K(x,y)|^2 = \sum_{y \in B(\kappa_m)} (\sum_{x \in A(\kappa_m)} |f_m^K(x,y)|^2)$. Now by using the Plancherel formula for A the inner sum can be rewritten so that we get $\sum_{y \in B(\kappa_m)} \#A(\kappa_m)^{-1} \sum_{\chi \in A(\kappa_m)^*} |\sum_{x \in A(\kappa_m)} f_m^K(x,y) \chi(-x)|^2$. By the Grothendieck-Lefschetz trace formula $\sum_{x \in A(\kappa_m)} f_m^K(x,y) \chi(-x) = f_m^L(y)$ holds for $L = Rp_*(K_\chi)$, hence

$$\|f_m^K\|_{X(\kappa_m)}^2 = \frac{1}{\#A(\kappa_m)} \sum_{\chi \in A(\kappa_m)^*} \sum_{y \in B(\kappa_m)} |f_m^{Rp_*(K_\chi)}(y)|^2 = \frac{1}{\#A(\kappa_m)} \sum_{\chi \in A(\kappa_m)^*} \|f_m^{Rp_*(K_\chi)}\|_{B(\kappa_m)}^2.$$

\square

13 Congruences and primary decomposition

Let k be the algebraic closure of a finite field κ of characteristic p . A perverse semisimple sheaf K on a variety X over k is a direct sum of certain irreducible perverse sheaves L . The support $Y = Y(L)$ of each L is an irreducible subvariety Y of X , and there exists a Zariski open dense subset $U \subseteq Y$ such that the restriction $L|_U$ of L to U is isomorphic to $E[\dim(U)]$ for some smooth local system E on U . The determinant $\det(E)$ defines a character of the étale fundamental group of U . We say that a semisimple perverse sheaf K on X satisfies the determinant condition, if $\det(E)$ is a character of finite order of the fundamental group of U for all the irreducible constituents L of K .

Proposition 2. *Let X be an abelian variety defined over κ . Let K a perverse semisimple Weil complex on X defined over κ which satisfies the determinant condition. If there exists an isomorphism $K * K \cong d \cdot K$ of Weil sheaves on X over κ , then K is a skyscraper sheaf on X .*

Proof. The condition $K * K \cong d \cdot K$ implies $f_m^{K * K} = f_m^K * f_m^K = d \cdot f_m^K$, hence

$$\widehat{f}_m(\chi)^2 = d \cdot \widehat{f}_m(\chi)$$

for all characters $\chi \in X_m^*$. Thus the Fourier transform \widehat{f}_m of f_m is constant and equal to d on its support.

By a theorem of Deligne [D] the determinant condition for K implies that there exists a number field E and with ring of integers \mathfrak{o} such that the functions f_m^K have their values in $\mathfrak{o}[\frac{1}{p}] \subset \Lambda$ for all m . For $\Lambda = \overline{\mathbb{Q}_l}$ and $l \neq p$ hence $f_m^K(x)$ has l -adic integral values and the same therefore holds for the Fourier transform \widehat{f}_m^K . For two characters $\chi', \chi \in X_m^*$ let us assume

$$\chi' = \chi \chi_l \quad , \quad (\chi_l)^{l^r} = 1$$

where $\chi_l \in X_m^*$ is a character of l -power order. Since the values of χ_l are l^r -th roots of unity and hence congruent to 1 in the residue field \mathbb{F} of $\mathfrak{o}[\zeta_{l^r}]$ modulo each prime ideal of $\mathfrak{o}[\zeta_{l^r}]$ over l , we get the congruence

$$\widehat{f}_m^K(\chi') \equiv \widehat{f}_m^K(\chi) \quad (\text{in } \mathbb{F}).$$

Indeed $\widehat{f}_m^K(\chi') = \sum_{x \in X_m} f_m^K(x) \chi'(-x) \equiv \sum_{x \in X_m} f_m^K(x) \chi(-x) = \widehat{f}_m^K(\chi)$, since $\chi'(-x) = \chi(-x) \chi_l(-x) \equiv \chi(-x)$ holds in \mathbb{F} . Now suppose that the prime l is larger than d and different from p . Then the congruence $\widehat{f}_m^K(\chi') \equiv \widehat{f}_m^K(\chi)$ implies

$$\widehat{f}_m^K(\chi') = \widehat{f}_m^K(\chi)$$

since \widehat{f}_m^K was equal to d on its support.

Now by a theorem of Drinfeld [Dr2] the following holds

Theorem 5. *Let X be a variety over κ and E be a finite extension of \mathbb{Q} . Let λ, λ' be nonarchimedean places of E prime to $p = \text{char}(\kappa)$ and E_λ and $E_{\lambda'}$ be the corresponding completions. Let K_0 be an irreducible \overline{E}_λ -adic perverse sheaf on X_0 so that for every closed point x in an open dense subset of its support $\det(1 - \text{Fr}_x t, K_{\overline{x}})$ has coefficients in E and its roots are λ -adic integers. Then there exists a $\overline{E}_{\lambda'}$ -adic perverse sheaf K'_0 on X compatible with K_0 (i.e. having the same characteristic polynomials of the Frobenius operators $\text{Fr}_{\overline{x}}$ for all closed points x in X).*

Proof. In the case where X is smooth and $K_0 = E_0[\dim(X)]$ is a smooth etale \overline{E}_λ -adic sheaf E_0 on X_0 this assertion is [Dr2, thm.1.1]. In general one applies Drinfeld's theorem for suitable smooth open Zariski dense subsets U_i of the support of the irreducible constituents of K_0 ; that the l -independence extends to the perverse intermediate extensions, i.e. to the constituents of K_0 , then is an easy exercise using purity arguments. Since this extension problem is a local question, one can assume that X is an affine variety and then by Noether normalization that X is affine space. Then the argument of [KW, p.164], combined with an easy purity argument left as an exercise, proves the claim. \square

By this theorem for every prime $\tilde{l} \neq p$ there always exists a $\overline{\mathbb{Q}}_{\tilde{l}}$ -adic semisimple perverse sheaf \tilde{K} so that $\tilde{\tau}(f_m^K) = \tau(f_m^K)$ holds for some isomorphisms $\tilde{\tau} : \overline{\mathbb{Q}}_{\tilde{l}} \cong \mathbb{C}$ and $\tau : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. So replacing l by some \tilde{l} larger than d and p we get, that \widehat{f}_m^K (which only depends on K and m and not on \tilde{l}) is translation invariant on the \tilde{l} -primary component $X_m^*[\tilde{l}^\infty]$ of the finite group X_m^* . Actually, since this holds for all primes l not dividing $N = p \cdot d$

$$\begin{array}{ccc} X(\kappa_m)^* & \xrightarrow{\widehat{f}_m^K} & E \\ & \searrow & \nearrow \exists \\ & \bigoplus_{l|p \cdot d} X(\kappa_m)^*[\tilde{l}^\infty] & \end{array}$$

Hence f_m^K as the Fourier inverse of \widehat{f}_m^K for all m must be a function with support contained in the N -primary subgroup of X_m

$$\text{supp}(f_m^K) \subseteq X_m[N^\infty] .$$

By the next lemma 17 this implies that K is a skyscraper sheaf. Indeed let Y be the support of an irreducible perverse constituent L of K . Then we claim $\dim(Y) = 0$. For this we may consider the constituents L for some Y for which $\dim(Y)$ is maximal. Then there exists a Zariski open dense subset $U \subset Y$ such that U is disjoint to the support of all irreducible components with support different from Y so that the restriction of the direct sum of irreducible components L of K to U becomes isomorphic to $E[\dim(U)]$ for some smooth etale sheaf E on U of rank say r . Assuming $E \neq 0$, then for any point $x \in U(k)$ there exists an integer m such that $x \in X_m$. Replacing m by a suitable multiple mn we find that $f_{mn}^K(x) = (-1)^{\dim(U)} f_{mn}^E(x) \neq 0$ holds, since otherwise the stalk $K_{\bar{x}}$ is zero contradicting the smoothness of $E \neq 0$ at x . This argument implies

$$U(k) = \bigcup_{m=1}^{\infty} U_m \subseteq \bigcup_{m=1}^{\infty} \text{supp}(f_m^K) \subseteq \bigcup_{m=1}^{\infty} X_m[N^\infty] .$$

Hence $\dim(U) = 0$ by lemma 17. In other words K is a skyscraper sheaf on X . \square

Lemma 17. *Let $U_0 \subseteq A_0$ be an open dense subset of an absolutely irreducible closed subset Y_0 of an abelian variety A_0 over a finite field κ . Suppose there exists an integer N such that*

$$U(\kappa_m) \subseteq A(\kappa_m)[N^\infty]$$

holds for all m . Then $\dim(Y) = 0$.

Proof. Y is projective, so for $\dim(Y) > 0$ there exists a curve in U defined over a finite field. If the assertion were false, without restriction of generality we may therefore assume $\dim(Y) = 1$. We may replace A by the abelian subvariety generated by the curve Y . For the irreducible closed subvariety $Y_n \subseteq A$ defined by the image of $a_n : Y^n \rightarrow A$ under the addition map $(y_1, \dots, y_n) \mapsto \sum_{i=1}^n y_i$ then $\dim(Y_n) \leq n$ holds for $n \leq d = \dim(A)$. In fact $\dim(Y) = n$ for $n \leq \dim(A)$, since otherwise some of these Y_n were stable under Y and hence stable under the simple abelian variety A generated by Y , contradicting $\dim(Y_n) \leq n$. Then $Y_d = A$ and $Y_{d-1} \neq A$ implies that the complement of $a_n(U^n)$ is contained in finitely many translates of Y_{d-1} , and that $a_n(U^n)$ therefore contains a Zariski open subset V . Replacing V by $V \cap -V$ as a set the union $V \cup (V + V) \cup (V + V + V) \cup \dots$ is A , since otherwise

there exists a U -invariant proper closed subset of A , which is impossible. By the noetherian property, then $A = \sum_{i=1}^m U$ for some m . Hence for all $a \in A$ there exist $y_i \in U$ for $i = 1, \dots, md$, such that $a = \sum_i y_i$. By our assumption $y_i \in A(k)[N^\infty]$ this implies $a \in A(k)[N^\infty]$ and therefore $A(k) = A(k)[N^\infty]$. This is impossible and shows $\dim(Y) = 0$. \square

We remark that obviously in the last lemma N could have been any supernatural number, provided it is not divisible by at least one prime number l .

14 Appendix

This appendix is not used in the previous sections. Also our topic is different, since we now discuss prime perverse sheaves K on products of simple abelian varieties over the algebraic closure k of a finite field κ , and this is motivated by [W] (for the notion prime perverse sheaf and also the definition of the invariant v_K we refer to [BN2]). For this let us in this section make the following assumptions:

- (A1) For simple abelian varieties X over k all perverse Weil sheaves K over finite subfields of k that are prime in the sense of [BN2] are translation invariant.
- (A2) If K is an acyclic perverse Weil sheaf on a simple abelian variety X , then each irreducible perverse constituent of K is acyclic (later applied for the perverse sheaves denoted M^i).

These two assumptions on simple abelian varieties over k are motivated by [KrW], where it is shown that the corresponding statements hold for simple abelian varieties over an algebraically closed field k of characteristic zero. We expect that the above assumptions unconditionally hold in the situation described above.

Using the assumptions (A1) and (A2) we show in corollary 7 that any irreducible perverse sheaf K with Euler characteristic zero on an arbitrary abelian variety X over k is translation invariant with respect to some nontrivial abelian subvariety of X . By the induction argument of [W, thm.4], for this it is enough to consider the case of irreducible prime perverse sheaves K with Euler characteristic

$\chi(K) = 0$ on products X of two simple abelian varieties A, B over k . In addition, as shown in [W, section 7], for the proof one can furthermore impose rather strict additional conditions on these prime perverse sheaves K , which naturally leads to conditions formulated preceding proposition 3. Using the arguments of [W, section 8], then our corollary 7 below is a consequence of the next proposition 3. The assumptions (A1) and (A2) are needed in order to verify that the rather technical conditions imposed on the prime perverse sheaf K in proposition 3 hold. In the characteristic zero situation the analog of the two assumptions (A1) and (A2) is always true [KrW]. Hence in the case $k = \mathbb{C}$ this allows us to apply proposition 3 using reduction mod p ; see [W, section 8]. In fact the assumptions of proposition 3 are inherited by the reduction, so that in this case it suffices to have (A1) and (A2) over \mathbb{C} so that conditions (A1) and (A2) are not needed for the field of reduction.

Although the results of this appendix do not depend on the results on the group of connected components given in the previous sections, for the proof of proposition 3 we use the same method as in the last section. Let us start with some

Notations. Assume $\Lambda = \overline{\mathbb{Q}}_l$ for some prime l different from p . Let κ be a finite field of characteristic p with q elements and κ_m be a finite extension field of κ of degree m with q^m elements. Let $F = F_\kappa$ denote the Frobenius automorphism of κ so that $F_{\kappa_m} = F^m$. Let X_0 be an abelian variety over κ and X its scalar extension to the algebraic closure k of κ . Let T_0 in $D_c^b(X_0, \Lambda)$ denote a semisimple translation invariant complex with the property ${}^p H^v(T_0) = 0$ for $v < \dim(X)$ and $v > -\dim(X)$. Let $H_0^\bullet = \bigoplus_{i=-d}^d H_0^i[-i]$ denote some fixed complex in $D_c^b(\text{Spec}(\kappa), \Lambda)$ and H^\bullet the associated F -module and $h_m := \text{Tr}(F^m; H^\bullet)$. We assume $\chi(H^\bullet) = \sum_v (-1)^v \dim_\Lambda(H^v) = 0$. Concerning these assumptions we remark that later we will apply this for $X_0 = A_0 \times B_0$ and $H^\bullet = H^\bullet(B, \delta_B)$ as a F_κ -module, related to the situation of [W, section 7]. Let $K_0 \in \text{Perv}(X_0, \Lambda)$ be a perverse sheaf, whose pullback K to X is a simple perverse sheaf on X . Let $\mathcal{S}(K)$ denote the spectrum of K , i.e. the set of ‘continuous’ characters $\chi : \pi_1(X, 0) \rightarrow \Lambda^*$ of the etale fundamental group of X for which $H^\bullet(X, K_\chi) \neq H^0(X, K_\chi)$ holds; in the following it suffices to consider torsion characters χ . For T we similarly define the spectrum $\mathcal{S}(T)$ of T . Let X_m denote the κ_m -rational points $X(\kappa_m)$ of X . Similarly $A_m = A(\kappa_m)$ and $B_m = B(\kappa_m)$. Let X_m^* be the group of characters $\chi : X_m \rightarrow \Lambda^*$, also considered as a group of characters of $\pi_1(X, 0)$ via Lang torsors as in section 11. This defines $\mathcal{S}_m(K) = \mathcal{S}(K) \cap X_m^*$ and ditto $\mathcal{S}_m(T)$ as the union $\bigcup_i \mathcal{S}({}^p H^i(T)) \cap X_m^*$. For a (super)natural number N let $X[N^\infty]$ denote the set of points in $X(k)$ annihilated by some power of N .

Define $\Lambda_0^* = \{\alpha \in \Lambda^* \mid |\alpha|_v = 1 \text{ for all archimedean and nonarchimedean valuation } |\cdot|_v \text{ not over } p\}$; by definition $\alpha \in \Lambda_{mot}^*$, if there exists integers n depending on α with $\alpha q^{n/2} \in \Lambda_0^*$ and we then write $w(\alpha) = -n$. Following [Dr2] a weakly motivic Weil complex K on X is a complex with the property that for all cohomology sheaves $\mathcal{H}^v(K)$ the eigenvalues of the Frobenius Fr_x on the stalks $\mathcal{H}^v(K)_{\bar{x}}$ at geometric points \bar{x} over closed points x of X have algebraic eigenvalues $\alpha \in \Lambda_{mot}^*$, i.e. there exists integers n depending on α with $\alpha q_x^{n/2} \in \Lambda_0^*$. By [Dr2, thm.B.3] the weakly motivic complexes define a thick triangulated full subcategory of $D_c^b(X_0, \Lambda)$. For k -morphisms $f : X \rightarrow Y$ these are stable under the functors $f_!, f_*, f^*, f^!, \boxtimes, D, R\mathcal{H}om$. For any weakly motivic complex the functions $f_m^K(x) = \sum_v (-1)^v Tr(F_\kappa^m; \mathcal{H}^v(K)_{\bar{x}})$ have values in a number field (independent from m) and these values are integral over $\mathbb{Z}[p^{-1}]$.

Now as in the next proposition assume that the irreducible perverse sheaf K satisfies

$$K^\vee \cong K \quad \text{and} \quad K^\vee * K \cong H^\bullet \cdot K \oplus T$$

(as shown in [W, prop.5] for prime perverse sheaves K on a product X of simple abelian varieties this automatically holds as a consequence of our assumption (A1) on simple abelian varieties). Replacing the irreducible Weil complex K by a generalized Tate twist $K(\alpha)$ for some $\alpha \in \Lambda^*$ we may achieve that the determinant of the smooth local coefficient system, defining K on some open dense subset of its support, has finite order [L]. Then K_0 is weakly motivic (see [Dr2]), and hence also H_0^\bullet and T_0 defined by

$$K_0 * K_0 \cong H_0^\bullet \cdot K_0 \oplus T_0 .$$

are weakly motivic as an immediate consequence. Notice $K_0^\vee \cong K_0(\alpha)$, and by the determinant condition α is a root of unity. Hence K is pure of weight 0. By suitably extending the finite field κ we may then assume $K_0^\vee \cong K_0$ so that

$$K_0^\vee \cong K_0 \quad \text{and} \quad K_0 * K_0 \cong H_0^\bullet \cdot K_0 \oplus T_0 .$$

Instead for the triple (K_0, H_0^\bullet, T_0) , this also holds for the triples $(K_{0\chi}, H_{0\chi}^\bullet, T_{0\chi})$ for the characters $\chi \in X_m^*$. The Euler characteristic of all K_χ is zero, since the Euler characteristic of H^\bullet and T is zero. Hence $\chi \in \mathcal{S}(K)$ iff $H^\bullet(X, K_\chi) \neq 0$, so that $\widehat{f}_m^K(\chi) = 0$ holds for all $\chi \notin \mathcal{S}(K)$ by the Grothendieck-Lefschetz formula (see page 40).

For $\chi \notin \mathcal{S}(T)$ then $H^\bullet(X, K_\chi)^{\otimes 2} \cong H^\bullet \otimes_\Lambda H^\bullet(X, K_\chi)$. Hence $\widehat{f}_m^K(\chi) \neq 0$ implies

$$\widehat{f}_m^K(\chi) = Tr(F^m; H^\bullet(X, K_\chi)) =: h_m ,$$

since there only two possible values for $\widehat{f}_m^K(\chi)$. We remark that $U \otimes_\Lambda W \cong V \otimes_\Lambda W$ and $W \neq 0$ for semisimple F -modules U, V, W implies $U \cong V$ after a finite field extension. Hence more precisely

$$\widehat{f}_m^K(\chi) = h_m \cdot 1_{\mathcal{S}_m(K)}(\chi) + \text{function supported in } \mathcal{S}(T)$$

holds for all m , after replacing κ by a suitable finite field extension. Hint: If the F -eigenvalue multi sets A, B, C of U, V, W satisfy $A + C = B + C$ in $C^* \otimes_{\mathbb{Z}} \mathbb{R}$, to show $A = B$ (as sets with multiplicities) simply use induction on $\dim(U) = \dim(V)$. For the induction step choose some arbitrary (lexicographic) ordering on the finite dimensional \mathbb{R} -vector space generated in $C^* \otimes_{\mathbb{Z}} \mathbb{R}$ by the elements in A, B, C that respects addition. Then compare $a + c$ and $b + c$ for the largest elements a, b, c in A, B, C to conclude $a = b$.

Now let $X_0 = A_0 \times B_0$ be a product of two simple abelian varieties A_0 and B_0 defined over κ . Let $p_0 : X_0 \rightarrow B_0$ be the projection on the second factor. The characters $\chi \in X_m^*$ correspond to pairs (χ_1, χ_2) of characters $\chi_1 \in A_m^*$ and $\chi_2 \in B_m^*$; we assume, that except for χ_1 in a finite set Σ , for every torsion character χ_1 there exist finitely many characters ψ_i depending on χ_1 such that $Rp_*(K_{\chi_1}) = \bigoplus_{i=1}^r m_i \cdot \delta_B^{\psi_i}$. For prime perverse sheaves this follows as in [W, lemma 9] from assumption (A1).

Then by [W, section 8] the multiplicities m_i are all one and the number $r = r(\chi_1)$ of characters appearing with multiplicity > 0 is independent of χ_1 and is called $\text{rank}(Rp_*(K))$. Up to a sign this is the Euler characteristic $r = (-1)^d \chi(M(b))$ of the complex $M(b) = K|_{A \times \{b\}}$ on $A \times \{b\}$. Attached to $M := M(0)$ we have the perverse sheaves $M^i = {}^p H^{-i}(M(0))$. Except for $i = 0, \dots, d = \dim(B)$ these M^i are zero. For $\chi_1 \notin \Sigma$ all M_χ^i are acyclic on $A = A \times \{0\}$ for $i \neq d$, and similarly for $M^i(b)$. By our second assumption (A2) all simple perverse constituents of these $M^i, i \neq d$ are acyclic and hence by the first assumption (A1) the $M^i, i \neq d$ then are translation invariant perverse sheaves on A . Notice K is pure of weight, so that $w(M) \leq 0$ and hence $w(M^i) \leq -i$ for all i .

Proposition 3. *For $X_0 = A_0 \times B_0$ and simple abelian varieties A_0, B_0 of dimension $d = \dim(A_0) = \dim(B_0)$ let K_0 be a simple prime perverse sheaf on X_0 with $v_K = d$ such that $K^\vee \cong K$ and $K^\vee * K \cong H^\bullet \cdot K \oplus T$ holds for a translation invariant complex $T = \bigoplus_{i=a}^b T_i[-i]$ on X with $-\dim(X) < a \leq b < \dim(X)$ and a graded Λ -vectorspace $H^\bullet = \bigoplus_{i=-d}^d H^i[-i]$ such that $H^{\pm d} \cong \Lambda$. Suppose that the support of K is not contained in a fiber of the projection $p_0 : X_0 \rightarrow B_0$ and that $\mathcal{H}^{-d}(K) = \delta_0$. Assume there exists a finite set Σ such that for all characters $\chi_1 \notin \Sigma$ of $\pi_1(A, 0)$ the direct image $Rp_*(K_{\chi_1})$ is a perverse translation invariant sheaf on B . Assume that all*

simple perverse constituents of these $M^i, i \neq d$ are translation invariant perverse sheaves on A . Then $M^d \cong \delta_0$ holds on $p^{-1}(0) = A \times \{0\}$ and in particular the rank r of the locally constant sheaf $Rp_*(K)$ is

$$\text{rank}(Rp_*(K)) = 1 .$$

Proof. Suppose $\chi = (\chi_1, \chi_2) \notin \mathcal{S}(T)$ and $\chi_1 \notin \Sigma$. This excludes finitely many characters χ_1 of $\pi_1(A, 0)$. Then by our assumptions

$$Rp_*(K_\chi) = \bigoplus_{i=1}^r \delta_B^{\psi_i}$$

with a multiplicity free collection of r characters ψ_i depending on χ (a special case of [BN2, prop. 2]). It suffices to consider $\chi = (\chi_1, 1)$ for torsion characters χ_1 , outside a finite set of exceptional characters.

Step 1) Under these assumptions on $\chi \in X_m^*$

$$Rp_{0*}(K_{0\chi}) \cong \bigoplus_j \text{Ind}_{\kappa_{mrj}}^{\kappa_m} (\delta_{B_0}^{\psi_j})(\alpha_j)$$

holds as an isomorphism of perverse Weil sheaves over κ_m in the sense of section 11 with α_j, r_j and ψ_j depending on χ . Since $K_{0\chi}$ is pure of weight 0, $Rp_{0*}(K_{0\chi})$ is pure of weight zero and hence $w(\alpha_j) = 0$ holds for all j .

Step 2) For $\chi \in A_m^*$, $b \in B_m$ and the subset $\{\psi_1, \dots, \psi_s\}$ of characters ψ_j from step 1) with the property $r_j = 1$ (i.e. $\psi_j \in B_m^*$) example 2 and 3 of section 11 show

$$f_m^{Rp, K_\chi}(b) = (-1)^d q_m^{-d/2} \cdot \sum_{j=1}^s \alpha_j^m \cdot \psi_j(b)^{-1} .$$

The number of summands $s = s_m(\chi)$ may depend on $\chi \in \mathcal{S}_m(K)$ and m . Also the twist factors α_j^m may a priori depend on $\chi \in \mathcal{S}_m(K)$.

Step 3) Notice that a twist of Rp_*K_χ with the inverse of $\psi_j = (1, \psi_j)$ for $j = 1, \dots, s$ gives $Rp_*K_{\chi/\psi_j} = (Rp_*K_\chi)_{\psi_j^{-1}} = \delta_B(\alpha_j) \oplus$ acyclic perverse sheaves on B . The twist factors $\alpha_j \in \Lambda_{mot}^*$ defining the ‘Tate twists’ in the last formula can therefore (for all m) be computed by the nonvanishing number

$$\begin{aligned} \alpha_j^m \cdot \text{Tr}(F_\kappa^m; H^\bullet(B, \delta_B)) &= \text{Tr}(F_\kappa^m; H^\bullet(B, \delta_B(\alpha_j))) \\ &= \text{Tr}(F_\kappa^m; H^\bullet(B, (Rp_*K_{\chi/\psi_j}))) = \text{Tr}(F_\kappa^m; H^\bullet(X, K_{\chi/\psi_j})) = \widehat{f}_m^K(\chi/\psi_j) . \end{aligned}$$

Now $\chi/\psi_j \in \mathcal{S}_m(K)$ follows from $\chi \in \mathcal{S}_m(K)$, and $\chi/\psi_j \notin \mathcal{S}_m(T)$ holds, since we discarded a finite set of exceptional characters $\chi = (\chi_1, 1)$. Hence as already shown

$$\widehat{f}_m^K(\chi/\psi_j) = h_m = \text{Tr}(F_\kappa^m; H^\bullet)$$

using $\chi/\psi_j \in \mathcal{S}_m(K)$. Thus the α_j^m are independent from j and independent from $\chi = \chi_1 \notin \Sigma$ in $\mathcal{S}_m(K)$. The arguments used to prove [W, prop. 5] also show that $\text{Tr}(F_\kappa^m; H^\bullet)$ is $\alpha^m \cdot \text{Tr}(F_\kappa^m; H^\bullet(B, \delta_B))$ for some twist factor α . Indeed the proof of corollary 7 of loc. cit. the factor H^\bullet is defined by the formulas $K * K \cong H^\bullet \otimes_\Lambda P \oplus T$ and $Rc_* c^*[d](P) = H^\bullet \otimes_\Lambda P$, which are identities of Weil sheaves and c over k comes from a homomorphism $c_0 : A_0 \times B_0^2 \rightarrow A_0 \times B_0$ defined over κ with kernel isomorphic to B_0 . Hence $H^\bullet = H^\bullet(B_0, \Lambda[d])$. Furthermore $P \cong K$ over k and hence $P_0 \cong K_0(\beta)$ for some β . This implies $H^\bullet = H^\bullet(B, \delta_B)(\alpha)$ for $\alpha = q^{d/2}\beta \in \Lambda_{\text{mot}}^*$. Notice $\text{Tr}(F_\kappa^m; H^\bullet(B, \delta_B)) = (-1)^d q_m^{-d/2} \#B_m \neq 0$ for all m , hence $\alpha_j^m = \alpha^m$. In particular $w(\alpha) = 0$ and $w(\beta) = -d$ holds for the weights and

$$f_m^{Rp_* K_\chi}(b) = (-1)^d \beta^m \cdot \sum_{j=1}^s \psi_j(b)^{-1}.$$

Step 4) K_χ and $Rp_*(K_\chi)$ being weakly motivic there exists a subring $\mathfrak{o} \subset \Lambda$ finite over \mathbb{Z}_l with prime ideal generated by π such that $f_m^{K_\chi}(x)$ and also $f_m^{Rp_*(K_\chi)}(b)$ have values in \mathfrak{o} . Notice \mathfrak{o} may depend on χ . For $\chi', \chi \in A_m^*$ and $\chi', \chi \notin \mathcal{S}(T)$ such that $\chi' = \chi \chi_l$ holds for an l -power torsion character χ_l from step 2) and 3) we get for all $b \in B_m$ the following equality in the residue field $\mathbb{F} = \mathfrak{o}/\pi$

$$f_m^{Rp_* K_{\chi'}}(b) \equiv f_m^{Rp_* K_\chi}(b).$$

This follows from $f_m^{Rp_*(K_{\chi'})}(b) = \sum_{a \in A_m} f_m^{K_{\chi'}}(a, b) = \sum_{a \in A_m} f_m^K(a, b) \cdot \chi'(a, b)^{-1}$ and the property $f_m^K(a, b) \in \mathfrak{o}$, using $\chi_l(a, b) \equiv 1 \pmod{\pi}$ and $\chi'(a, b)^{-1} \equiv \chi(a, b)^{-1}$ in \mathbb{F} .

Step 5) Writing $f_m^{Rp_* K_\chi}(b)$ as a finite linear combination of s characters $\psi_j : B_m \rightarrow \mathbb{F}^*$ as in step 2), 3), we get for $b = 0$

$$f_m^{Rp_* K_\chi}(0) = (-1)^d \cdot \beta^m \cdot s$$

where the number of characters $s = s_m(\chi)$ may depend on χ and m . By the congruence formula with $\chi' = \chi \chi_l$ as in step 4) then

$$\beta^m \cdot \sum_{i=1}^{s(\chi')} \psi'_j(b)^{-1} \equiv \beta^m \cdot \sum_{i=1}^{s(\chi)} \psi_j(b)^{-1},$$

which holds for all $b \in B_m$. Hence $\beta \in \Lambda_{mot}^*$ (so that $\beta \neq 0$ in \mathbb{F}) and the linear independency of characters $B_m \rightarrow \mathbb{F}^*$ implies the congruence $s_m(\chi') \equiv s_m(\chi)$. This gives an equality $s_m(\chi') = s_m(\chi)$, if l is larger than the Euler characteristic r of $M(0)$. For this recall that r is an upper bound for $s = s_m(\chi)$ that is independent of χ and m . Hence for $l > r$ we get an equality

$$f_m^{Rp_*K_{\chi'}}(0) = (-1)^d \cdot \beta^m \cdot s_m(\chi') = (-1)^d \cdot \beta^m \cdot s_m(\chi) = f_m^{Rp_*K_{\chi}}(0)$$

for $\chi' = \chi\chi_l$ provided the conditions of step 4) are satisfied.

Step 6) By theorem 5 this can be applied for any primes $l' \neq p$ and $l' > r$ instead of the given prime l . Let $N = p \prod_{l' \leq r} l'$ be the product of the finitely many remaining primes. Then $\chi \mapsto f_m^{Rp_*(K_{\chi})}(0)$ (for all m) is an $A_m^*[l'^{\infty}]$ -invariant function on A_m^* for the primes $l' \nmid N$, at least on the complement of the set $\mathcal{S}_m(T)$. In fact these functions (for varying m) can be extended over $\mathcal{S}_m(T)$ to functions called $\widehat{f}_m(\chi)$ on all of A_m^* that are simultaneously $A_m^*[l'^{\infty}]$ -invariant for all primes $l' \nmid N$. Indeed for $\chi \in \mathcal{S}_m(T)$ extend by comparing χ with $\chi\chi_l \notin \mathcal{S}_m(T)$ respectively $\chi\chi_{l'} \notin \mathcal{S}_m(T)$ etc.; this patches together in a well defined way. To see this we compare with characters $\chi\chi_l\chi_{l'} \notin \mathcal{S}_m(T)$ for some m large enough.

Step 7) By the last step for $\chi = (\chi_1, 1)$ we get for the extensions $\widehat{f}_m(\chi)$ defined in step 6) the formula

$$\widehat{f}_m^M(\chi_1) = f_m^{Rp_*(K_{\chi})}(0) = \widehat{f}_m^K(\chi) = \widehat{f}_m(\chi) + \text{function with support in } \mathcal{S}_m(T)$$

for all $\chi_1 \in A_m^*$ and all m .

Step 8) The perverse sheaves M^i for $i \neq d$ are translation invariant on A . Hence in the formula of step 7) we can replace M by M^d after replacing $\mathcal{S}(T)$ by a suitable larger finite set \mathcal{S} of characters (independent of m). By Fourier inversion then for all $a \in A_m$

$$f_m^{M^d}(a) = f_m(a) + g_m(a) \quad , \quad g_m(a) = \sum_{\chi \in \mathcal{S}} g(m, \chi) \cdot f_m^{\delta_A^{\chi}}(a)$$

holds for the Fourier inverse $f_m(a)$ of $\widehat{f}_m(\chi)$. Notice that by step 6) the function f_m has support

$$\text{supp}(f_m) \subseteq A_m[N^{\infty}]$$

for all m . By enlarging κ we may assume that all characters in \mathcal{S} are in $A(\kappa)^*$, and by enlarging N that also $\chi^N = 1$ holds for $\chi \in \mathcal{S}$. This implies $g_m(a+x) = g_m(a)$

for all x in some subgroup U_m of $A(\kappa_m)$ with index dividing $N^{2\dim(X)}$. In particular, this holds for all points $x \in A(\kappa_m)[l^\infty]$ for primes l not dividing N . Now fix such a prime l .

Step 9) The semisimplification $(M^d)^{ss}$ of the perverse sheaf M^d on the simple abelian variety A can be written as a sum $(M^d)^{ss} = T \oplus \delta$, where T is translation invariant and where δ is clean in the sense that it does not contain translation invariant summands. Let Y be an irreducible component of maximal dimension of the support $\text{supp}(\delta)$ of δ . Choose a Zariski open subset $Y' \subset Y$ which is disjoint to other irreducible components of $\text{supp}(\delta)$ and so that $\delta|_{Y'}$ is smooth. The Zariski closure of $A(k)[l^\infty]$ in A is A . For $\delta \neq 0$ there exists an integer m and a point $x \in A(\kappa_m)[l^\infty]$ such that $Y' + x \notin \text{supp}(\delta)$. Therefore we can find $m_0, a \in Y'(\kappa_{m_0})$ so that $f_m^\delta(a+x) = 0$ holds for all m and so that $f_m^\delta(a) \neq 0$ holds for infinitely many m . In the Grothendieck group of perverse sheaves on A then the linear combination

$$T_x^*((M^d)^{ss}) - (M^d)^{ss} = T_x^*(\delta) - \delta$$

is nontrivial, with certain components having generic support of dimension $\dim(Y)$ not cancelling away. On the other hand by step 8), for varying m the associated functions of this element in the Grothendieck group are $a \mapsto f_m^{M^d}(a+x) - f^{M^d}(a) = f_m(a+x) + g_m(a+x) - f_m(a) - g_m(a) = f_m(a+x) - f_m(a)$, and their support hence is contained in $A(\kappa_m)[(lN)^\infty]$.

Step 10) The last step shows that for all m the functions

$$f_m^{T_x^*(\delta)}(a) - f_m^\delta(a)$$

have their support in $A_m[(lN)^\infty]$. If we evaluate these functions at points of general position of the support, this yields a contradiction unless $\dim(\text{supp}(\delta)) = 0$. This follows from the Cebotarev density theorem [Dr2, lemma 1.5] and lemma 17 (applied for lN instead of N). Hence the semisimplification $(M^d)^{ss}$ of the perverse sheaf M^d on A is

$$(M^d)^{ss} \cong T \oplus \delta,$$

for some perverse Weil sheaf δ on A with support of dimension 0 and some translation invariant perverse Weil sheaf T_A on A .

Step 11) By the last step $\delta \cong \bigoplus_i m_i \cdot \delta_{x_i}(\beta_i)$ for finitely many closed points $x_i \in A$ with certain multiplicities $m_i \geq 1$ and twists $\beta_i \in \Lambda_{\text{mot}}^*$. We choose κ large enough so that the points x_i are κ -rational points.

Step 12) From step 9), 10), 11) for all m and for almost all characters $\chi = \chi_1$ we get

$$f_m^{Rp_*(K_\chi)}(0) = \sum_{a \in A_m} f_m^{M_\chi}(a) = \sum_{a \in A_m} f_m^{M^d}(a) = \sum_{a \in A_m} f_m^{\delta_\chi}(a) = \sum_i m_i \cdot \beta_i^m \cdot \chi(x_i)^{-1}.$$

Step 13) For pointwise weights we have $w(M^d) \leq w(M) - d \leq w(K) - d = -d$. We claim that the weights of the stalks of δ at the points x_i in the support are equal to $-d$; as a skyscraper sheaf then δ is pure of weight $-d$. Indeed by comparing step 11) and step 5) for all m and all $\chi \in A_m^*$ (with finitely many exceptions) we get

$$(-1)^d \cdot s_m(\chi) = \sum_i m_i \cdot \theta_i^m \cdot \chi(x_i)^{-1}$$

for the numbers $\theta_i = \beta_i/\beta$ in Λ_{mot}^* . Hence by regrouping for pairwise different θ_k this shows

$$(-1)^d \cdot s_m(\chi) = \sum_k \theta_k^m \cdot m_k(\chi) \quad , \quad m_k(\chi) = \sum_i m_{ki} \cdot \chi(x_i)^{-1} \neq 0$$

with integer coefficients $m_{ki} \geq 0$. Since $w(\beta) = -d$ and $w(\beta_i) \leq -d$ and $s_m(\chi)$ is an integer, if $w(\beta_i) < -d$ would hold for one of the β_i , this would give a contradiction for $m \rightarrow \infty$. Indeed this would imply $w(\theta_k) < 0$ for some k . Then choose $\chi \in A_m^*$ such that $\chi(x_i) = 1$ holds for all i and for some $m = m_0$. We can also assume $s_{m_0}(\chi) = r$ by enlarging m_0 if necessary. Then for $m = vm_0$ and $v \rightarrow \infty$ we have $s_m(\chi) = r$, and the formulas above easily give a contradiction. This proves $w(\beta_i) = -d$ for all i .

Step 14) Since $w(M^d) \leq -d$ and since δ is pure of highest weight $-d$ as shown in step 13), the weight filtration on M^d and the decomposition theorem imply the existence of an exact sequence

$$0 \rightarrow T \rightarrow M^d \rightarrow \delta \rightarrow 0$$

of perverse sheaves on A .

Step 15) By our assumption $\mathcal{H}^{-d}(K) \cong \delta_0$ the perverse sheaf K can not be translation invariant under A . This implies $M^d \cong \delta$, since the existence of a translation invariant perverse subsheaf $T \neq 0$ in M^d would imply that K is translation invariant under A using the argument of step 2 and 3 in the proof of [W, prop.2] (the argument in loc. cit. does not depend on the choice of the base point b).

Step 16) K is an irreducible prime perverse sheaf with $\mathcal{H}^{-d}(K)_0 \cong \Lambda$ by assumption. The stalk spectral sequence of [W, section 1] before lemma 2 of loc. cit. therefore gives an exact sequence of étale sheaves

$$0 \longrightarrow \mathcal{H}^{-d}(M^1) \xrightarrow{\partial_d} \mathcal{H}^0(M^d) \longrightarrow \mathcal{H}^{-d}(K)|_{A \times \{0\}} \longrightarrow \mathcal{H}^{-d}(M_0) \ .$$

Indeed the higher differentials $\partial_i : \mathcal{H}^{-i}(M^j) \rightarrow \mathcal{H}^0(M^d)$ for $i + j = d + 1$ and $i = 2, \dots, d - 1$ all vanish, because the perverse sheaves M^i for $i < d$ are translation invariant on A and hence their cohomology sheaves are zero except in degree $-d$. Furthermore $M^0 = 0$ holds by our assumptions on the support of K_0 . Therefore $\partial_d = 0$ and $\mathcal{H}^0(M^d)_a \cong \mathcal{H}^{-d}(K)_a$ holds for all $a \in A \times \{0\}$, since $w(\mathcal{H}^{-d}(M^1)) \leq -d - 1$ and since M^d is a pure skyscraper sheaf of weight w . Thus

$$\delta = \mathcal{H}^{-d}(K)|_{A \times \{0\}} \ .$$

Since $v_K = d$, by [BN2, lemma 1, part 5 and 7] the support of $\mathcal{H}^{-d}(K)$ can be identified with the finite stabilizer group of the prime perverse sheaf K and each stalk has Λ -dimension one.

Step 17) By our assumptions on $\mathcal{H}^{-d}(K)$ therefore $\delta = \delta_0$. In particular the Euler characteristic of $Rp_*(K)$ or M is $(-1)^d$, since $\chi(M^d) = \chi(\delta_0) = 1$. Hence $\text{rank}(Rp_*(K)) = (-1)^d \cdot \chi(M) = 1$. \square

Remark. Applying proposition 3 as in the proof of [W, thm. 5], the roles of A and B can be interchanged. Using this, then step 3 of our proof implies $\#A_m = c^m \cdot h_m = \#B_m$ for almost all m . Since $\sum_{m \geq m_0} \#A_m t^m$ determines the L -function of A , and similar for B , we get $\text{Tr}(F_\kappa^m, H^1(A, \mathbb{Q}_l)) = \text{Tr}(F_\kappa^m, H^1(B, \mathbb{Q}_l))$ for all m . Hence A is isogenous to B by the Tate conjectures for abelian varieties over finite fields. If A and B are isogenous, for the proof of corollary 7 over finite fields using the reductions to products of simple abelian varieties as in [W], we can assume $A = B$ and the abstract subgroup $\bigcup_{n,m \in \mathbb{Z}} \{(na, ma) \mid a \in A\}$ is Zariski dense in $X = A \times A$. So for every proper closed subset $D \subset X$ there exists n, m with $(n, m) = 1$ such that $\varphi_M(A \times \{0\}) \cap D$ is a proper closed subset of $\varphi_M(A \times \{0\})$ for the automorphism $\varphi_M \in \text{Aut}(X)$ defined by a unimodular matrix $M \in \text{GL}(2, \mathbb{Z})$ with entries $M_{11} = n$ and $M_{21} = m$. If the support of K is X , then $\text{supp}(\mathcal{H}^{-2 \dim(A)}(K)) = X$ and $\mathcal{H}^{-2 \dim(A)}(K)$ is a smooth nonvanishing sheaf on $U = X \setminus Y$ for some Zariski closed subset $Y \subset D$ with D of dimension $d - 1$. The intersection $U \cap \varphi_M(A \times \{0\})$ is Zariski dense in $\varphi_M(A \times \{0\})$ unless $\varphi_M(A \times \{0\}) \subset Y$. Hence without restriction of generality

$M^d = {}^p H^{-\dim(A)}(K|_{A \times \{0\}})$ can be assumed to have support $A \times \{0\}$ by replacing K with $\varphi_M^*(K)$ for some M .

This however contradicts step 15 of our last proof. In the characteristic zero case this proves the analog of corollary 7 below in view of [W, prop.4]. Another argument in characteristic zero relying on proposition 3 is given in [W, prop.6]. Besides the assumptions (A1) and (A2) this is the only place where in the proofs of [W] the characteristic zero assumption enters, namely by using the support condition $\text{supp}(K) = X$ in the proof of [W, prop.6]. That this support condition holds in characteristic zero follows from [W, thm.3] and its corollary [W, prop.4].

Let us now give a third argument via Lang torsors, which completely avoids the use of the support condition $\text{supp}(K) = X$ made in [W, prop.6] and therefore, in view of the results of loc. cit., implies corollary 7 over finite fields:

Let K_0 respectively K be perverse sheaves as in the last proposition 3. We want to apply this proposition to the pullback $\mathcal{I}_0^{(m)*}(K_0)$ of K_0 with respect to some Lang morphism $\mathcal{I}_0^{(m)} : X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m) \rightarrow X_0 \times_{\text{Spec}(\kappa)} \text{Spec}(\kappa_m)$ respectively the corresponding isogeny over k

$$0 \longrightarrow X(\kappa_m) \longrightarrow X \xrightarrow{\mathcal{I}_0^{(m)}} X \longrightarrow 0 .$$

For this we replace κ by κ_m and K by $\tilde{K} = \mathcal{I}_*^{(m)}(K)$. Notice \tilde{K} is defined over κ_m and we write \tilde{K}_0 for the corresponding perverse sheaf on $X_0 \times_{\kappa} \kappa_m$. By [BN2, cor. 4] the perverse sheaf \tilde{K} is a simple prime perverse sheaf with the same support as K so that $\mathcal{H}^{-d}(\tilde{K}) = \delta_0$. So once we replace X_0, A_0, B_0 by their scalar extension to κ_m the perverse sheaf \tilde{K} satisfies all the conditions required for proposition 3. Since $\mathcal{I}_*^{(m)}$ is a tensor functor, the properties $\tilde{K}^\vee \cong \tilde{K}$ and $\tilde{K} * \tilde{K} \cong H^\bullet \cdot \tilde{K} \oplus \tilde{T}$ hold for the translation invariant complex $\tilde{T} = \mathcal{I}_*^{(m)}(T)$ on B , since the corresponding property holds for K . Finally

$$Rp_*(\tilde{K}) = \mathcal{I}_*^{(m)}(Rp_*(K)) ,$$

since p is defined over κ and therefore commutes with $\mathcal{I}_*^{(m)}$. Similarly for characters $\varphi = \chi \circ \pi_1(\mathcal{I}_*^{(m)})$ we get $Rp_*(\tilde{K}_\chi) = \mathcal{I}_*^{(m)}(Rp_*(K_\varphi))$. Hence $Rp_*(\tilde{K}_{\chi_1})$ is a translation invariant perverse sheaf on B for almost all characters χ_1 , since it is the direct image of the corresponding translation invariant perverse sheaf $Rp_*(K_\varphi)$ for $\varphi = (\varphi_1, \varphi_2)$ and $\varphi_1 \notin \Sigma$. Therefore proposition 3 can be applied for \tilde{K} instead of K and again shows

$$\text{rank}(Rp_*(\tilde{K})) = 1 .$$

On the other hand also $\text{rank}(Rp_*(K)) = 1$ by proposition 3. Twisting K we may assume that $Rp_*(K)$ is perverse, so that $Rp_*(K) \cong \delta_B^\Psi$. Hence we get

$$\text{rank}(Rp_*(\tilde{K})) = \text{rank}(\mathcal{P}_*^{(m)}(Rp_*(K))) = \deg(\mathcal{P}^{(m)}) \cdot \text{rank}(Rp_*(K)) = \deg(\mathcal{P}^{(m)})$$

from $\mathcal{P}_*^{(m)}(\delta_B^\Psi) = \bigoplus_{i=1}^{\deg(\mathcal{P}^{(m)})} (\delta_B^\Psi)_{\chi_i}$. This implies $\#X(\kappa_m) = \deg(\mathcal{P}^{(m)}) = 1$, which is absurd for m large enough. This contradiction shows that perverse sheaves K with the properties required in proposition 3 do not exist. Together with the arguments of [W] (especially from loc. cit. section 5,7 and 8) this implies

Corollary 7. *Let k be the algebraic closure of a finite field κ . Suppose any prime perverse Weil sheaf on a simple abelian variety over k is translation invariant. Furthermore suppose that all simple constituents of an acyclic perverse sheaf on a simple abelian variety are acyclic. Then any irreducible perverse Weil sheaf with Euler characteristic zero on an arbitrary abelian variety X over k is translation invariant under some abelian subvariety $Y \subseteq X$ of positive dimension.*

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